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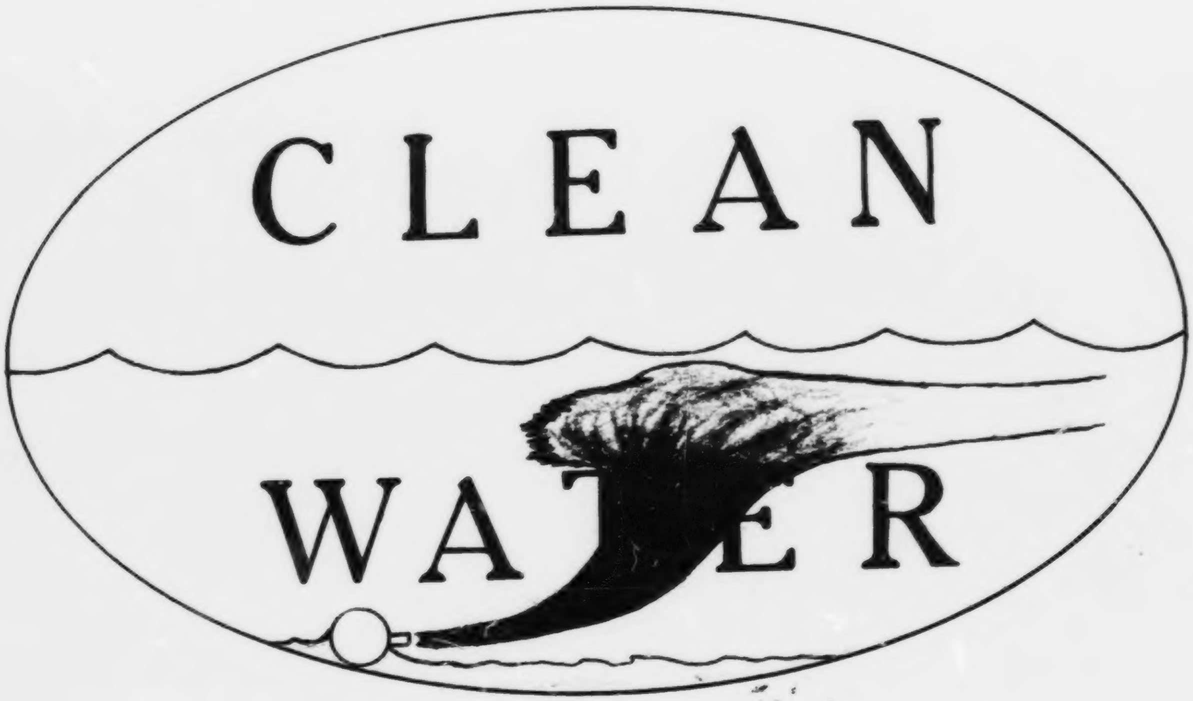

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The growth of population and industry in the coastal regions of the country, along with increased use of coastal waters for disposal of wastes produced inland, presents the potential of new or increased pollution in a valuable national resource. In order to assure that marine water quality is maintained and enhanced for present and future water uses, each source of waste input must be evaluated for its impact on the environment. Treated wastes must be discharged in such a manner and location that the resulting concentrations and contact times do not violate the water quality requirements specified for the desired water uses. The disposal process can be designed within these constraints through a mix of at least four techniques: degree and type of waste treatment; site selection; outfall or barge distribution of wastes; barrier construction. ~~This~~ The report presents a review of the literature pertinent to the theoretical development of present ocean outfall design technology as it applies to waste discharges in general. (Gruchalla-EPA)

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OCEAN OUTFALL DESIGN
PART I
Literature Review and Theoretical Development

by

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April 1970

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INTRODUCTION

The growth of population and industry in the coastal regions of the country, along with increased use of coastal waters for disposal of wastes produced inland, presents the potential of new or increased pollution in a valuable national resource. In order to assure that marine water quality is maintained and enhanced for present and future water uses, each source of waste input must be evaluated for its impact on the environment. Treated wastes must be discharged in such a manner and location that the resulting concentrations and contact times do not violate the water quality requirements specified for the desired water uses. The disposal process can be designed within these constraints through a mix of at least four techniques:

- degree and type of waste treatment;
- site selection;
- outfall or barge distribution of wastes;
- barrier construction.

This report presents a review of the literature pertinent to the theoretical development of present ocean outfall design technology as it applies to waste discharges in general. Part II, entitled "Recommended Procedures for Design and Siting," is in preparation and will incorporate references and data collection methods for specific pollutants.

Barge discharge of wastes is becoming more important because of the increasing production of concentrated wastes, and the increased concern for continued incineration and ground disposal of wastes presently practiced inland. This technology will be explored in a future report.

POLLUTANTS

Material pollutants are those commonly associated with municipal and industrial wastes: biological species, chemicals, sludge, etc. Another material pollutant, which has caused no problem in the past but which promises some cause for concern in the future, is the highly saline reject water from desalinization plants. A non-material pollutant new to coastal waters is appearing on the scene, threatening to present problems in the near future which might surpass those of municipal and industrial wastes. This pollutant is heat - reject heat from very large thermal power stations using coastal water for condenser coolant. In either case we need to be able to predict pollutant concentrations in the local sea, as a function of both location and time.

Modeling Pollution Fields Through Mathematics

The problem of mathematically modeling pollutant dispersion in coastal waters involves solving the basic equations of marine hydrodynamics, and heat (or mass) transfer. Although the fundamental equations which describe the differential motion and diffusion are reasonably well understood, solution of these equations is analytically intractable without gross simplification of the oceanic motion, pollutant flow, and boundary conditions. In addition, the terms associated with pollutant decay and interaction are frequently not well described.

Analytical research, then, has been a process of fashioning simplifications in order to derive tractable mathematical models. Along with the analytical research, laboratory and field experiments have yielded additional information which developed reliable semi-empirical models.

An approach to the problem of mathematical modeling which has apparently not yet been fully utilized is the use of numerical techniques in conjunction with high-speed digital computation. The numerical approach involves approximating the governing differential equation with finite quantities and carrying out the required numerical computations on a digital computer. The advantage of this procedure is that many complications that must be ignored in the semi-analytic models can be incorporated in the numerical model. Regardless of which approach is taken, certain data must be obtained in the field. Generally, mean currents, some estimation of turbulent transport coefficients, density stratification, and pollutant interaction data are needed.

Subsequent discussion in this report will be restricted to only those characteristics of a pollutant which influence the general description of its fate, such as density. The considerations of decay, interaction, sedimentation, and air-sea exchange, etc., which might be unique for any specific pollutant, are not included.

THE NATURE OF POLLUTION PLUMES IN OCEANIC ENVIRONMENTS

Before investigating theoretical aspects and the various methods of analyzing the dispersion of pollutants emitted from ocean outfall systems, the gross behavior of the pollution flow field will be reviewed to give a general idea of the nature of the flow.

Fluid which issues continuously from an outfall pipe on the ocean floor passes through several flow regimes as it disperses into the surrounding ocean. It may contain pollutants in the form of either heat or matter; and, when ejected into the surrounding environment, it may be buoyant or not. However, unless specifically stated otherwise, discussion will be concerned with a positively buoyant fluid. It is common to refer to free convection flow in the environment as a plume, for in this case the general pattern of motion is caused by virtue of a density disparity between the fluid and the surroundings. A jet is characterized by initial momentum of the fluid. If the injected flow has a density disparity and momentum, then the flow assumes some character of both types of flow. This mixed flow could be called either a buoyant jet or a forced plume. Since the overall pollution dispersion in the sea resembles a plume more than a jet, the term forced plume or plume will be used here.

The Regimes of Flow

Experimental investigations of forced plumes have revealed the existence of three distinct flow regimes, as follows (see Figure 1):

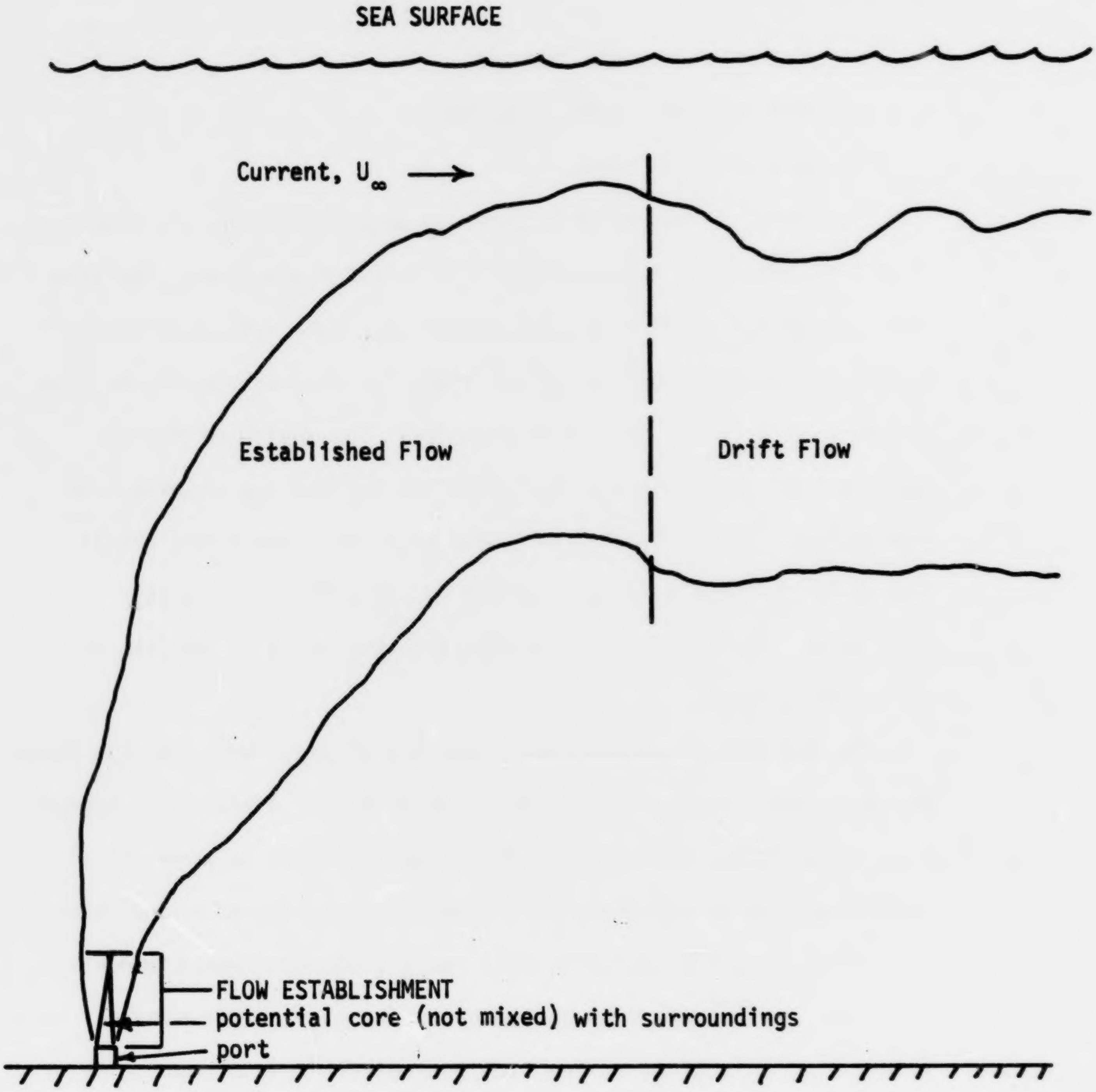
SEA SURFACE
Current, U_∞
Established Flow
Drift Flow
FLOW ESTABLISHMENT
potential core (not mixed) with surroundings
port

Figure 1. Regions of flow for fluid issuing into a density-stratified medium with current.

- zone of flow establishment,
- zone of established flow, and
- zone of drift flow.

The zone of flow establishment is associated with the transition from pipe flow to an established forced plume flow. The flow from an outfall orifice or pipe enters the ocean with essentially a uniform velocity profile, which begins to deteriorate at the flow boundary as a result of turbulent shear. The region of mixing spreads both inward toward the center of the jet and outward into the medium. Within a short distance from the orifice the interchange of momentum will have spread to the center of the jet. At this point, it is generally assumed that the velocity profile is fully established.

In the zone of established flow, the driving force for the plume may be either momentum or buoyancy, or both. As distance is increased from the orifice, the width of the plume increases as a result of lateral mixing or turbulent diffusion (commonly called entrainment).

Eventually the pollutant will reach a position where momentum no longer affects the flow and buoyancy may play only a minor role. In this region, called 'the zone of drift flow' here, the spread of the pollutant is dominated by the environmental turbulence, although a lateral density flow may also be present if the pollution field is situated at the surface with residual buoyancy.

Environmental Effects

The environment into which the fluid is discharged can influence the nature of the plume flow, due to

- density stratification,
- environmental velocity (oceanic currents),
- environmental turbulence,
- air-sea interactions.

Density Stratification

Both theory and experiment have shown that a buoyant plume will reach the surface if the ambient fluid is homogeneous with respect to density.

However, the ocean is rarely homogeneous in this respect, except perhaps in shallow coastal waters where wave action causes good vertical mixing to occur. Where the ocean is density stratified, it is possible that the plume will not penetrate to the surface, at least in the zone of established flow. The reason for this phenomena is as follows.

The plume entrains the heaviest environmental fluid near the orifice, decreasing the density disparity between the plume and surroundings. As the plume ascends the decrease continues as the density of the surroundings decreases. The plume will eventually reach a point of neutral buoyancy, which may be below the surface. At this point the flow continues upward only because of the vertical momentum it possesses. As the flow continues upward, it continues to entrain liquid, but the plume is heavier than the surroundings; hence, the flow is negatively buoyant. Eventually, all upward vertical momentum is lost and, since the plume liquid is heavier than the surroundings, the pollutant will cascade downward around the upward flow until it reaches equilibrium with the surroundings.

The general shapes of plumes in non-stratified and stratified stagnant liquids are shown below.

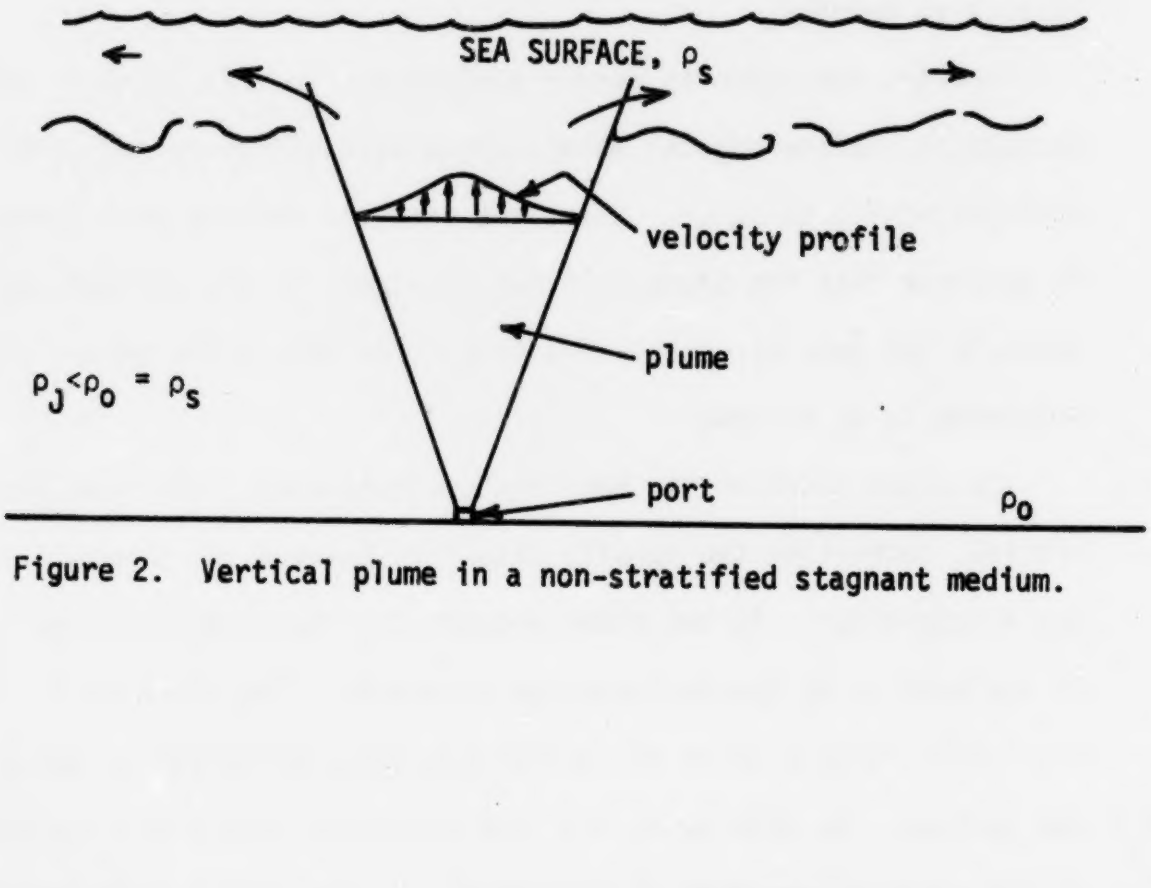
SEA SURFACE, ρ_s
velocity profile
plume
$\rho_J < \rho_o = \rho_s$
port
ρ_o

Figure 2. Vertical plume in a non-stratified stagnant medium.

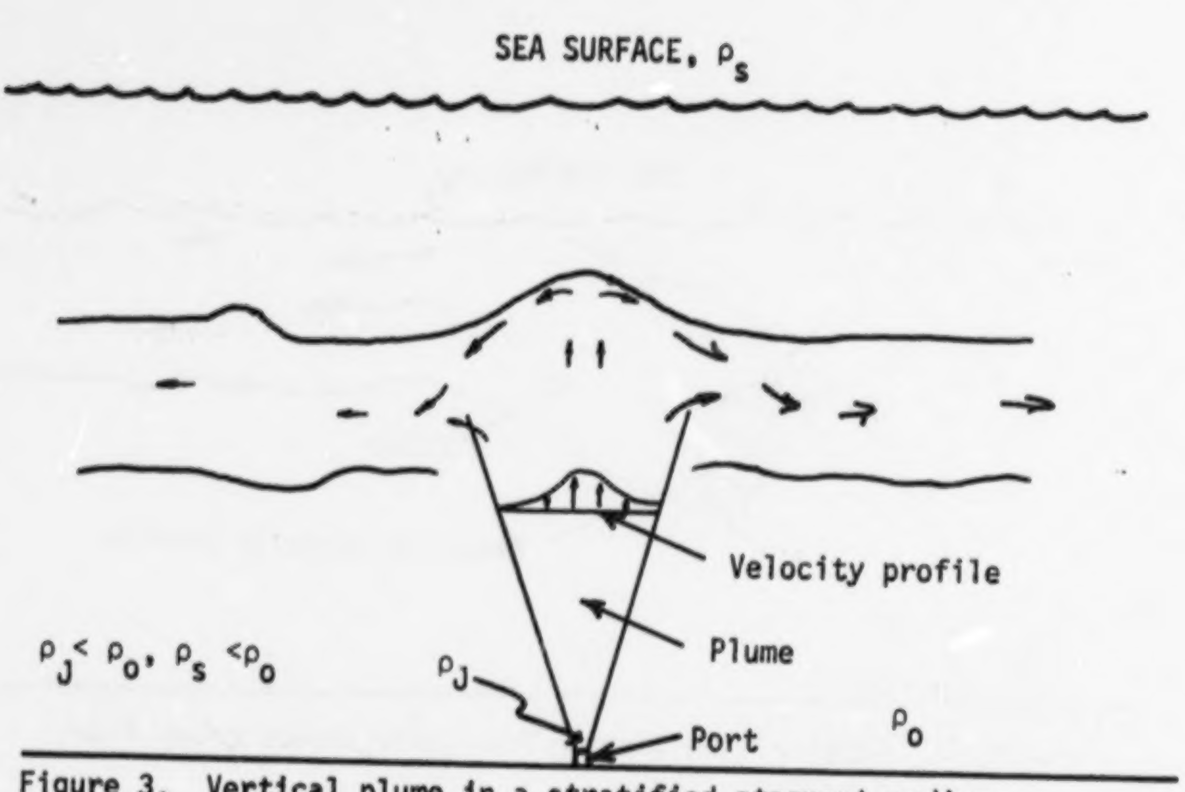
SEA SURFACE, ρ_s
Velocity profile
Plume
$\rho_J < \rho_o$, $\rho_s < \rho_o$
ρ_J
Port
ρ_o

Figure 3. Vertical plume in a stratified stagnant medium.

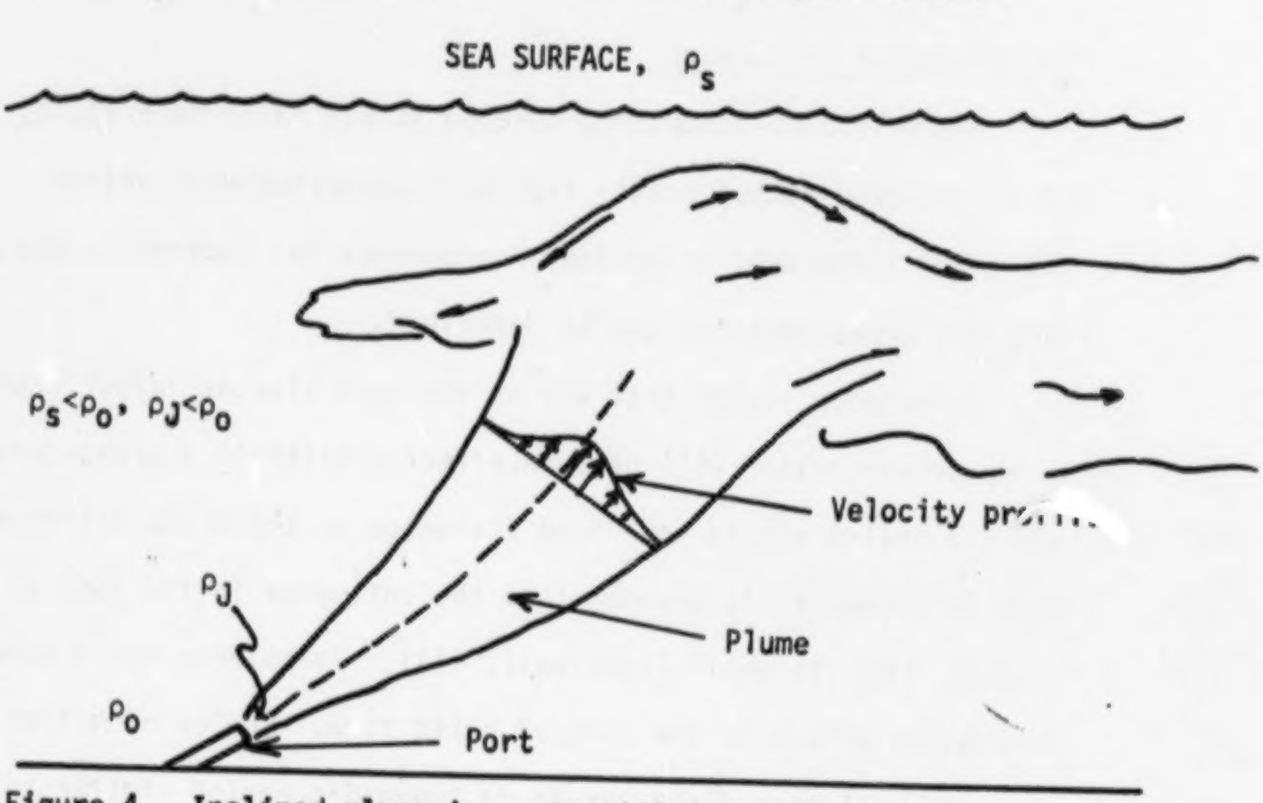
SEA SURFACE, ρ_s
$\rho_s < \rho_o$, $\rho_J < \rho_o$
Velocity pr
Plume
ρ_J
ρ_o
Port

Figure 4. Inclined plume in a stratified stagnant medium.

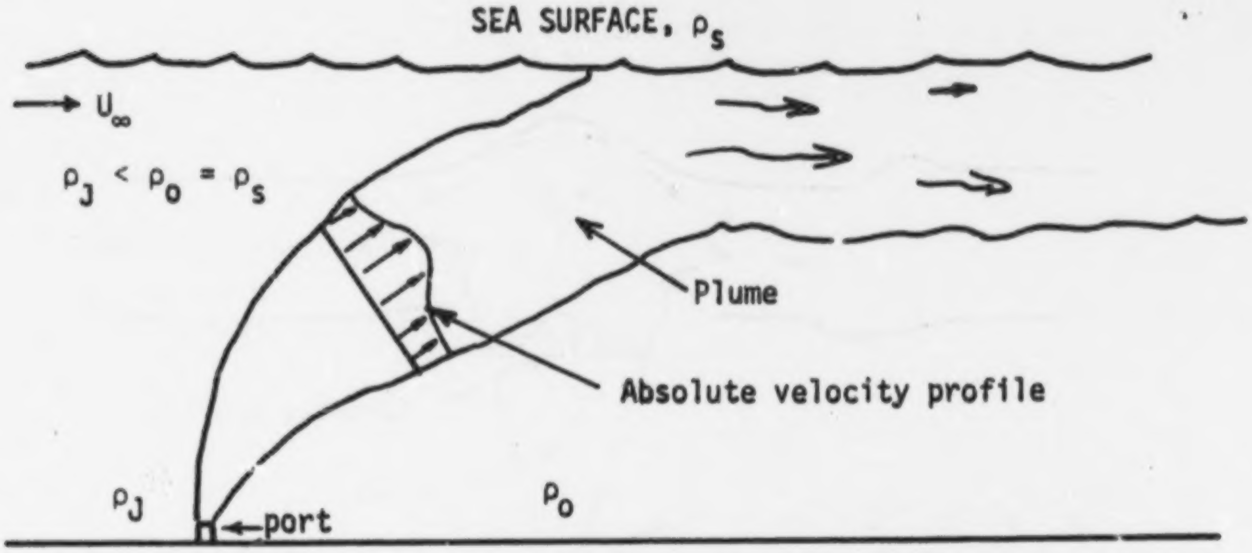

Figure 5. Vertical jet issuing into homogeneous cross flow.

Environmental Turbulence

The origin of oceanic turbulence is not fully understood, although it is probably caused mostly from wind-generated wave action. As such, the turbulence is neither homogeneous nor isotropic, and only the gross behavior can be described.

Turbulence scales that are on the same size or larger than the plume cross-section will have an effect similar to a cross-current, and all scales should have some influence on the plume entrainment rate (although it is thought that the influence in the zone of established flow is small (Cederwall, 11)). Turbulence has a more dominating effect in the zone of drift flow. Scales of motion larger than the pollutant field result in transport action similar to oceanic currents, while smaller scales of motion add to the eddy diffusion of the pollutant; thus, as the pollutant field spreads, larger and larger scales of eddy mixing come into play.

Since most oceanic waters are density stratified (except perhaps near the surf zone), vertical turbulence is suppressed to a great extent. Thus, a pollution field may disperse much more rapidly in the lateral direction than in the vertical direction.

Air-sea Interactions

Although wind stress can indirectly affect the motion and dilution of a submerged pollution field through perturbing the mean oceanic motion and by causing turbulence, the most dramatic interactions with the atmosphere occur when the pollution field develops on the surface. The two major interactions are wind stress and transfer of heat and mass.

Wind-driven surface currents may cause the surface drifting pollutant to stagnate along the coast or cause it to be carried out to sea. Transfer at the sea surface is of primary importance in the case of thermal pollution. Thermal energy interchange at the sea surface will occur through the mechanisms of convection, evaporation, and radiation. Volatilization and oxidation of oil and other pollutants are also important considerations.

Outfall Geometry

Outfall geometry also affects the nature of the plume. Greater dilution can be achieved with outfall structures employing a series of horizontal rather than vertical ports or orifices. In this case, each plume will follow a curved path until it reaches either the maximum height of rise or the sea surface. Port spacing is important because of the possibility of plume interaction and the resulting decrease in dilution efficiency. Figure 6 illustrates a possible flow configuration from a ported outfall with a slight downstream drift of the density-stratified receiving water. Unless otherwise indicated, discussions here will be confined to round orifices.

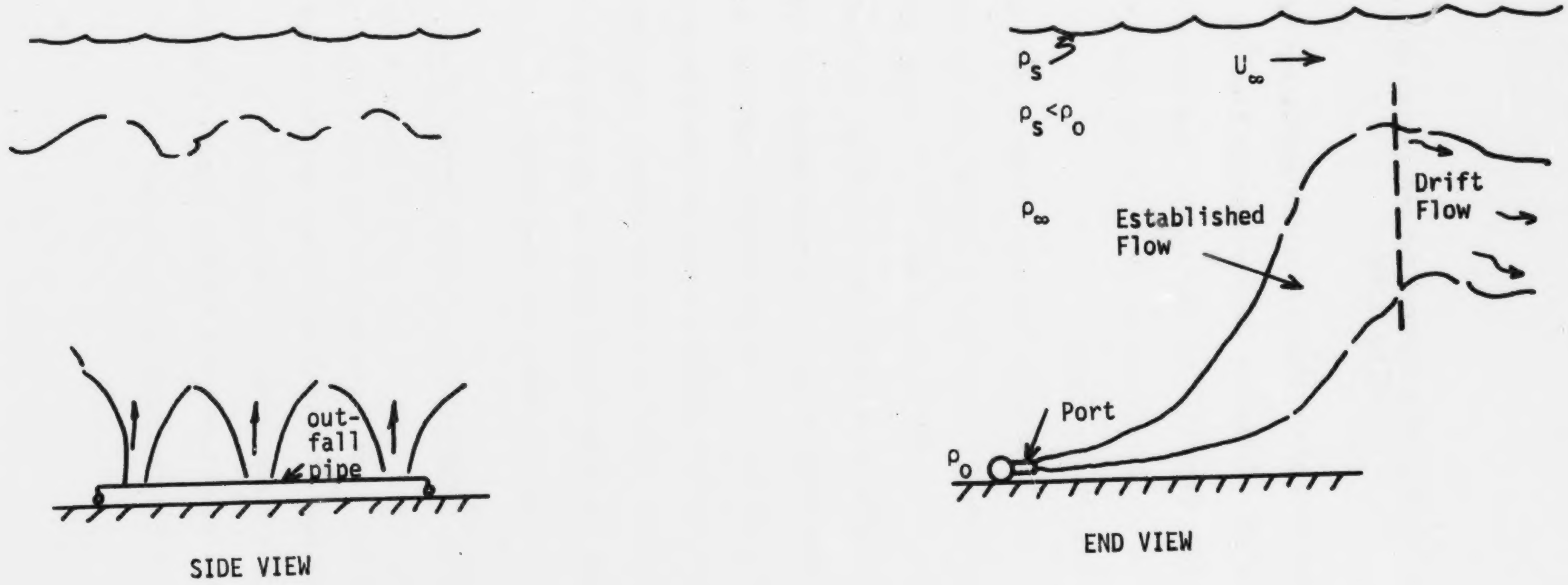
out-
fall
pipe
SIDE VIEW
ρ_s
$\rho_s < \rho_o$
ρ_∞
U_∞
Established
Flow
Drift
Flow
Port
ρ_o
END VIEW

Figure 6. Interacting plumes from a ported outfall line with horizontal discharge, in a density-stratified environment.

THE FUNDAMENTAL EQUATIONS GOVERNING THE MOTION AND DILUTION OF A FORCED PLUME

Prior to assessment of various analytical approaches for evaluating plume dynamic behavior, the fundamental laws and equations which govern the flow will be introduced. The reason for this introduction is to clarify the physical grounds upon which plume mathematical models are based.

There are two basic approaches one may follow in deriving plume models - one is based on a differential analysis, and the other on an integral analysis. The differential approach involves solving the general partial differential equations of motion and heat (or mass) diffusion to arrive at velocity and temperature (or concentration) distributions. In the case of an integral analysis, specific shape of the profiles must be assumed, based on intuition or experiment. Since the integral analysis involves several simplifications and assumptions about the flow field, analytic discussions of them will be deferred until special cases are considered.

The Differential Equations for a Forced Thermal Plume

To be specific, the discussion here will be based on the consideration of a thermal plume where heat is the pollutant. Where desirable, analogous equations will be written and discussions presented for the case where matter is the pollutant.

The differential equations for a thermal plume are derived from consideration of the following physical laws:

- Continuity (conservation of mass,
- Newton's Second Law (conservation of momentum),
- First Law of Thermodynamics (conservation of energy), and
- An Equation of Volumetric Expansion.

Detailed derivation of the equations of motion and energy will not be given here, since these equations can be found in nearly usable form in texts dealing with fluid mechanics or convective heat transfer (e.g. Ref. 7). A few modifications must be made for application to a plume in an oceanic environment. The coordinate system for the following discussion is shown in Figure 7.

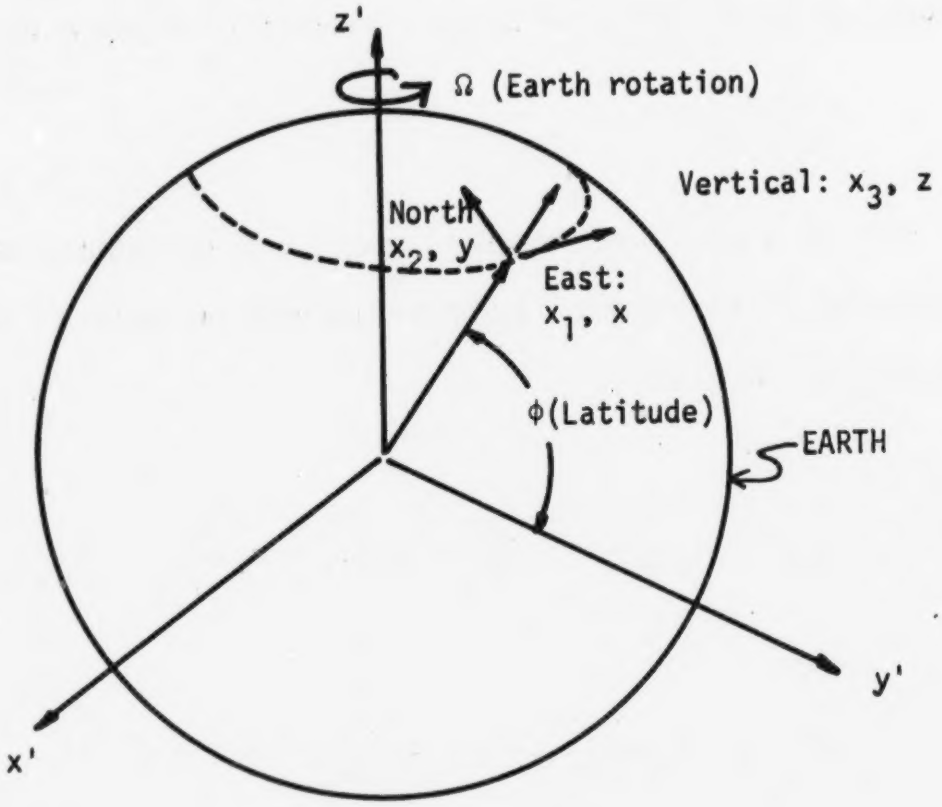
z'
Ω (Earth rotation)
Vertical: x_3, z
North x_2, y
East: x_1, x
ϕ(Latitude)
EARTH
y'
x'

Figure 7. Rectangular coordinate system.

The differential equations for incompressible, density stratified laminar flow written in Cartesian tensoral form are as follows:

Continuity:

$$\frac{\partial \rho}{\partial t} + U_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial U_i}{\partial x_i} = 0.$$

In the ocean, density varies with both time and space as indicated by the above equation. However, changes caused by temperature or mass concentration variations are usually quite small and may be neglected in consideration of mass conservation. Thus, the continuity equation may be approximated with acceptable accuracy by

$$\frac{\partial U_i}{\partial x_i} = 0. \tag{1}$$

It will be shown later, however, that these variations cannot be neglected in the momentum equation and are the point of concern in the energy equation.

Momentum:

$$\rho\left(\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j}\right) = - \frac{\partial P}{\partial x_i} - g\rho\delta_{i3} + \mu\frac{\partial^2 U_i}{\partial x_j \partial x_j} . \tag{2}$$

Energy:

$$\rho C_p \frac{\partial T}{\partial t} + C_p \rho U_j \frac{\partial T}{\partial x_j} = \kappa \frac{\partial^2 T}{\partial x_j \partial x_j} + \Phi + \dot{q} .^{*} \tag{3}$$

Volumetric Expansion:

$$\beta = \frac{1}{v}\frac{\partial v}{\partial T} = - \frac{1}{\rho}\frac{\partial \rho}{\partial T} . \tag{4}$$

---

* The molecular fluid properties μ and k are assumed isotropic in this work.

In the above equations the Einsteinian notation is used (a repeated index in a term implies summation over all three indexes--1, 2, and 3).

The symbol δ_{i3} is the Kronecker delta and is equal to 1 when $i = 3$, and otherwise zero. The term Φ which appears in the energy equation is a viscous dissipation term and may be neglected. For simplicity, changes in ρ are assumed a function of T alone in equation (4).

Consider the fluid density expressed as $\rho = \rho_\infty(x_3) + \Delta\rho$, where $\rho_\infty(x_3)$ is the reference density.*

Equation 2 may be written as:

$$\rho\left(\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j}\right) = -\frac{\partial P}{\partial x_i} - (\rho_\infty + \Delta\rho) g\delta_{i3} + \mu \frac{\partial^2 U_i}{\partial x_j \partial x_j} .$$

Let ρ_o be a constant reference density, specifically $\rho_o = \rho_\infty(0)$.

Division by ρ_o, noting that $(\rho_o \doteq \rho)$ yields

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho_o} \frac{\partial P}{\partial x_i} + \left(\frac{\rho_\infty - \rho}{\rho_o} - 1\right) g\delta_{i3} + \nu \frac{\partial^2 U_i}{\partial x_j \partial x_j} . \quad (5)$$

This manipulation was done to separate out the buoyancy effect

$$\left(\frac{\rho_\infty - \rho}{\rho_o}\right) g\delta_{i3} .$$

The important idea to note here is that $\Delta\rho$ has significance in establishing the above buoyancy term in the momentum equation;

---

* Hereafter, the reference density $\rho_\infty(x_3)$ will be written simply as ρ_∞ with dependence on coordinate x_3 always implied.

however, ignoring small variations of ρ will not otherwise affect the outcome significantly (the so-called "Boussinesq approximation"). The symbol ν indicates kinematic viscosity and is defined as $\nu = \mu/\rho$.

Equations (3) and (5) must now be modified for application to the oceanic environment. First of all, the equations of motion must be modified to account for the earth's rotation (the Coriolis effect). If Ω is the earth's rotation vector, equation (5) becomes

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + 2e_{ijk}\Omega_j U_k = -\frac{1}{\rho_o}\frac{\partial P}{\partial x_i} + \left(\frac{\rho_\infty - \rho}{\rho_o} - 1\right) g\delta_{i3} + \nu \frac{\partial^2 U_i}{\partial x_j \partial x_j} ,$$

where e_{ijk} is the permutation tensor which takes values of zero if any two of the three subscripts i, j and k are identical, +1 for even permutations, and -1 for odd permutations of the subscripts.

Next the equations of motion and the energy equation must be modified to account for oceanic turbulence. To do this the dependent variables are considered as composed of a mean or average part and a superimposed random fluctuating part; for instance,

$$U_i = \bar{U}_i + u_i ,$$

where $\bar{U}_i$ is the mean velocity and u_i is the fluctuation part. These definitions are substituted into the equations of motion, and the result is time averaged term by term over a sufficiently long period of time.

The resulting equations of motion for turbulent flow are then

$$\frac{\partial \bar{U}_i}{\partial t} + \bar{U}_j \frac{\partial \bar{U}_i}{\partial x_j} + \frac{\partial}{\partial x_j}(\overline{u_i u_j}) + 2e_{ijk}\Omega_j \bar{U}_k =$$

$$- \frac{1}{\rho_o}\frac{\partial \bar{P}}{\partial x_i} + \left(\frac{\rho_\infty - \bar{\rho}}{\rho_o} - 1\right) g\delta_{i3} + \nu \frac{\partial^2 \bar{U}_i}{\partial x_j \partial x_j} , \qquad (6)$$

which is seen to be identical in the mean motion with equation (5) except for the appearance of the term

$$\frac{\partial}{\partial x_j}(\overline{u_i u_j}).$$

A new quantity is now defined:

$$R_{ij} = -\bar{\rho}\, \overline{u_i u_j},$$

which is called the Reynolds stress. Finally the complete equations of motion in the rotating earth reference frame are written as

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + 2e_{ijk}\Omega_j U_k = - \frac{1}{\rho_o}\frac{\partial P}{\partial x_i}$$

$$+ \left(\frac{\rho_\infty - \rho}{\rho_o} - 1\right) g\delta_{i3} + \frac{\nu \partial^2 U_i}{\partial x_j \partial x_j} + \frac{1}{\rho_o}\frac{\partial R_{ij}}{\partial x_j} , \qquad (7)$$

for the mean flow. Here the overbars denoting average quantities have been omitted. Another way in which the turbulent stress terms are handled is through the Prandtl mixing length theory (e.g. Ref. 36). In this case

$$R_{ij} = -\bar{\rho}\, \overline{u_i u_j} = \rho_o \varepsilon_{ij} \frac{\partial U_i}{\partial x_j} ,$$

so that the equations of motion become:

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + 2e_{ijk}\Omega_j U_k = -\frac{1}{\rho_o}\frac{\partial P}{\partial x_i}$$

$$+ \left(\frac{\rho_\infty - \rho}{\rho_o} - 1\right) g\delta_{i3} + \frac{\partial}{\partial x_j}\left[(\nu + \epsilon_{\underline{ij}})\frac{\partial U_i}{\partial x_j}\right], ^{*} \qquad (8)$$

where again variations in ρ are considered small and ϵ_{ij} is the eddy diffusion coefficient for momentum, a second order tensor.

The turbulent form of the energy equation may be derived in a similar fashion. For constant ρ and C_p the energy equation for laminar flow is written as

$$\frac{\partial T}{\partial t} + U_j \frac{\partial T}{\partial x_j} = \frac{\partial}{\partial x_j}\left[a \frac{\partial T}{\partial x_j}\right] + \frac{\dot{q}}{\rho C_p} ,$$

where a is the molecular thermal diffusivity, $\kappa/\rho C_p$. Accounting for turbulent heat transport yields

$$\frac{\partial T}{\partial t} + U_j \frac{\partial T}{\partial x_j} = \frac{\partial}{\partial x_j}\left[(a + \epsilon_{H\underline{j}})\frac{\partial T}{\partial x_j}\right] + \frac{\dot{q}}{\rho C_p} , \qquad (9)$$

where ϵ_{Hj} is defined as the coefficient of eddy heat diffusion, a first order tensor, and may be directionally biased, whereas the fluid is assumed to be isotropic with respect to a. The analogous form of equation (9) for the case of diffusion of matter is:

$$\frac{\partial C}{\partial t} + U_j \frac{\partial C}{\partial x_j} = \frac{\partial}{\partial x_j}\left[(\mathcal{D} + \epsilon_{M\underline{j}})\frac{\partial C}{\partial x_j}\right] + \dot{m}, \qquad (10)$$

---

* The summation convention for repeated tensoral indices does not apply to underscored indices in this text.

where C is the pollutant concentration, ε_{Mj} is the eddy diffusivity of mass, $\mathcal{D}$ is the molecular diffusion coefficient, and $\dot{m}$ is the production or loss of the species mass per unit volume. Equation (10) is based on the diffusion of a single species whose concentration is $C(x_1, x_2, x_3, t)$. These equations, (1), (4), (8), and (9) (or (10)), then form a complete set of equations from which a plume may be analyzed for dynamic behavior. Unfortunately, the equations are extremely complex and evidently cannot be solved analytically for the general case.

Rectangular Coordinates

For analysis of the local sea, the geopotential surface may be assumed flat. Rectangular coordinates are defined in Figure 7. The Coriolis term in equation (8) may be expanded, as:

$$2e_{ijk}\Omega_j U_k = 2\begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ 0 & \Omega\text{Cos}\phi & \Omega\text{Sin}\phi \\ U & V & W \end{vmatrix}, \tag{11}$$

so that:

$$2e_{ijk}\Omega_j U_k = 2\Omega(W\ \text{Cos}\phi - V\ \text{Sin}\phi)\vec{e}_x + U\ \text{Sin}\phi\ \vec{e}_y$$

$$- U\ \text{Cos}\phi\ \vec{e}_z\ , \tag{12}$$

where $\vec{e}_x$, $\vec{e}_y$, and $\vec{e}_z$ are unit vectors for the east, north, and vertical directions, respectively. Then in component form the governing equations become:

Continuity:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0. \tag{13}$$

x-Momentum:

$$\frac{DV}{Dt} + 2\Omega(W \, \text{Cos}\phi - V \, \text{Sin}\phi) = -\frac{1}{\rho_0}\frac{\partial P}{\partial x}$$

$$+ \frac{\partial}{\partial x}\left[(\nu+\varepsilon_{xx})\frac{\partial U}{\partial x}\right] + \frac{\partial}{\partial y}\left[(\nu+\varepsilon_{xy})\frac{\partial U}{\partial y}\right] + \frac{\partial}{\partial z}\left[(\nu+\varepsilon_{xz})\frac{\partial U}{\partial z}\right].^{*} \tag{14}$$

y-Momentum:

$$\frac{DV}{Dt} + 2\Omega U \, \text{Sin}\phi = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + \frac{\partial}{\partial x}\left[(\nu+\varepsilon_{yx})\frac{\partial V}{\partial x}\right]$$

$$+ \frac{\partial}{\partial y}\left[(\nu+\varepsilon_{yy})\frac{\partial V}{\partial y}\right] + \frac{\partial}{\partial z}\left[(\nu+\varepsilon_{yz})\frac{\partial V}{\partial z}\right]. \tag{15}$$

z-Momentum:

$$\frac{DW}{Dt} - 2\,\Omega U \, \text{Cos}\phi = -\frac{1}{\rho_0}\frac{\partial P}{\partial z} + \left[\frac{\rho_\infty - \rho}{\rho_0} - 1\right] g + \frac{\partial}{\partial x}\left[(\nu+\varepsilon_{zx})\frac{\partial W}{\partial x}\right]$$

$$+ \frac{\partial}{\partial y}\left[(\nu+\varepsilon_{zy})\frac{\partial W}{\partial y}\right] + \frac{\partial}{\partial z}\left[(\nu+\varepsilon_{zz})\frac{\partial W}{\partial z}\right]. \tag{16}$$

---

* The operator, $\frac{D}{Dt} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y} + W\frac{\partial}{\partial z}$, is known as the substantial derivative.

Energy:

$$\frac{DT}{Dt} = \frac{\partial}{\partial x}\left[(\alpha+\varepsilon_{Hx})\ \frac{\partial T}{\partial x}\right] + \frac{\partial}{\partial y}\left[(\alpha+\varepsilon_{Hy})\ \frac{\partial T}{\partial y}\right]$$

$$+ \frac{\partial}{\partial z}\left[(\alpha+\varepsilon_{Hz})\ \frac{\partial T}{\partial z}\right] + \frac{\dot{q}}{\rho C_p}\ . \tag{17}$$

Cylindrical Coordinates

Figure 8 below illustrates the cylindrical coordinate system.

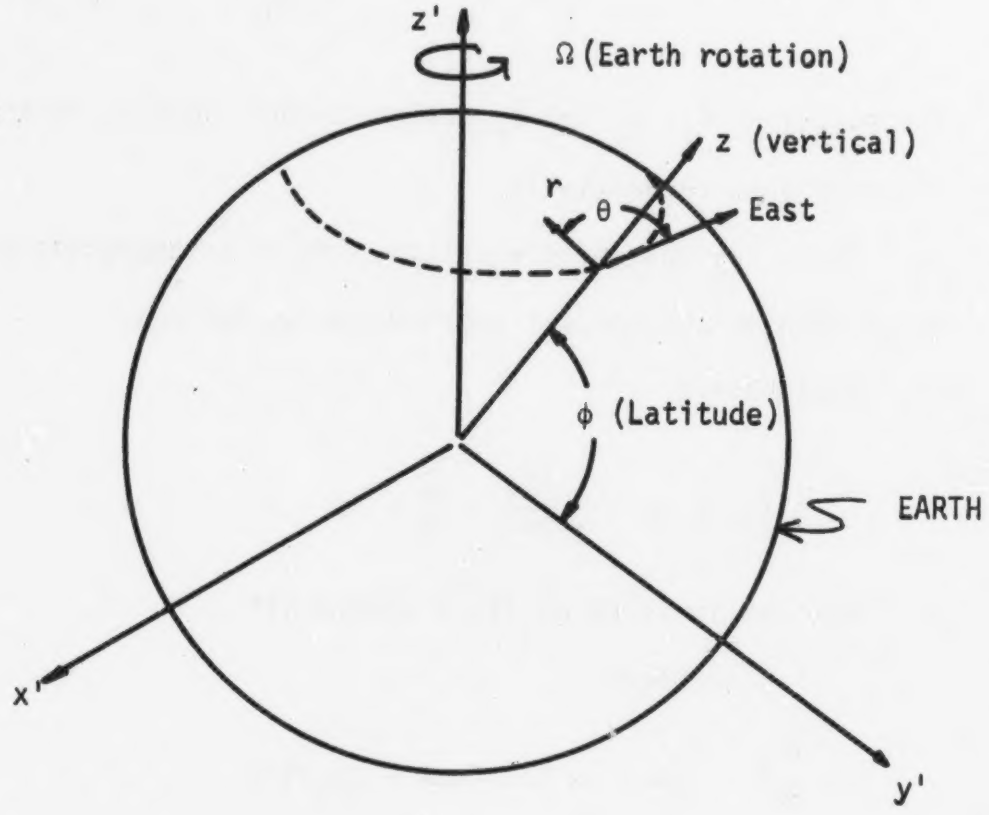
z'
Ω (Earth rotation)
z (vertical)
r
θ
East
ϕ (Latitude)
EARTH
x'
y'

Figure 8. Cylindrical coordinate system.

In this system, the z-direction is perpendicular to the geopotential surface, ϕ is the latitude, r is a rotating vector which lies in the flat geopotential plane at an angle θ measured from east, and Ω is again the earth's rotational velocity.

The coriolis components in cylindrical coordinates are given by:

$$2e_{ijk}\Omega_j U_k = 2 \begin{vmatrix} \vec{e}_r & \vec{e}_\theta & \vec{e}_z \\ \Omega \mathrm{Cos}\phi \mathrm{Sin}\theta & \Omega \mathrm{Cos}\phi \mathrm{Cos}\theta & \Omega \mathrm{Sin}\phi \\ U_r & U_\theta & W \end{vmatrix} \tag{18}$$

The notations $\vec{e}_r$, $\vec{e}_\theta$ and $\vec{e}_z$ designate unit vectors in the r, θ and z directions, respectively.

Thus, the governing equations for an incompressible flow may be written in cylindrical coordinates as follows:

Continuity:

$$\frac{1}{r}\frac{\partial}{\partial r}(U_r r) + \frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{\partial W}{\partial z} = 0. \tag{19}$$

Motion (in terms of fluid stress τ)*

r-direction:

$$\frac{DU_r}{Dt} - \frac{U_\theta{}^2}{r} + \Omega W \,\mathrm{Cos}\phi \mathrm{Cos}\theta - \Omega U_\theta \mathrm{Sin}\phi$$

$$= -\frac{1}{\rho_o}\frac{\partial P}{\partial r} - \frac{1}{\rho_o}\left(\frac{1}{r}\frac{\partial r\tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{rz}}{\partial z}\right)^{**} \tag{20}$$

---

* The equations of motion in terms of stress τ are more appropriate for development to be considered later.

** In cylindrical coordinates, $\frac{D}{Dt} = \frac{\partial}{\partial t} + U_r \frac{\partial}{\partial r} + \frac{U_\theta}{r}\frac{\partial}{\partial \theta} + W\frac{\partial}{\partial z}$.

θ-direction:

$$\frac{DU_\theta}{Dt} + \frac{U_r U_\theta}{r} + \Omega U_r \mathrm{Sin}\phi - \Omega W \mathrm{Cos}\phi \mathrm{Sin}\theta$$

$$= -\frac{1}{\rho_o r}\frac{\partial P}{\partial \theta} - \frac{1}{\rho_o}\left(\frac{1}{r^2}\frac{\partial(r^2\tau_{\theta r})}{\partial_r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\theta z}\right). \qquad (21)$$

z-direction:

$$\frac{DW}{Dt} + \Omega U_\theta \mathrm{Cos}\phi \mathrm{Sin}\theta - \Omega U_r \mathrm{Cos}\phi \mathrm{Cos}\theta$$

$$= -\frac{1}{\rho_o}\frac{\partial P}{\partial z} + \left(\frac{\rho_\infty - \rho}{\rho_o} - 1\right)g - \frac{1}{\rho_o}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{zr}) + \frac{1}{r}\frac{\partial \tau_{z\theta}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z}\right). \qquad (22)$$

Energy:

$$\frac{DT}{Dt} = \frac{1}{r}\frac{\partial}{\partial r}\left((a+\epsilon_{Hr})r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left((a+\epsilon_{H\theta})\frac{\partial T}{\partial \theta}\right)$$

$$+ \frac{\partial}{\partial z}\left((a+\epsilon_{Hz})\frac{\partial T}{\partial z}\right) + \frac{\dot{q}}{\rho C_p}. \qquad (23)$$

Integration of the Governing Differential Equations

The differential equations governing the motion and dilution of a thermal plume were set down in equations (1), (8), and (9). The problem of predicting pollutant dispersion, therefore, lies in gaining the solution to these equations, once having established such requirements as boundary conditions, eddy transport coefficients, etc. However, these equations are evidently impossible to solve analytically while maintaining all terms of the equations required to describe the motion and dilution of the pollution plume from its source to its ultimate fate.

Simplified Hydrodynamic Equation

Only a very few analytical solutions to the equations of motion (8) are known and these have come by accepting gross simplifications of the physical phenomena. For instance, gross motion in the deep oceans as influenced by Coriolis effects (earth rotation), sea surface slope, density anomalies, etc., fall into a class of motion called "frictionless flow," implying the frictional terms in the equations of motion have been neglected. In other simplified flows, frictional terms are included under specific conditions. Also, vertical momentum is usually ignored. Examples of frictionless flow are as noted:

- Inertial Currents -

$$\frac{dU}{dt} = 2\Omega V \mathrm{Sin}\phi \tag{24}$$

$$\frac{dV}{dt} = -2\Omega U \mathrm{Sin}\phi . \tag{25}$$

The term $2\Omega W \mathrm{Cos}\phi$ has been neglected in equation (24), since it is small compared to $2V\Omega \mathrm{Sin}\phi$. Equations (24) and (25) are used to approximate the motion of a water mass during an acceleration where horizontal pressure forces are comparatively small.

- Geostrophic Currents -

For a water mass in steady motion, Coriolis forces are balanced by lateral pressure forces, so that:

$$2V\Omega \mathrm{Sin}\phi = \frac{1}{\rho}\frac{\partial P}{\partial x} \tag{26}$$

$$2U\Omega \mathrm{Sin}\phi = -\frac{1}{\rho}\frac{\partial P}{\partial y} . \tag{27}$$

Gradient Currents:

If inertial terms are important for a water mass in steady motion, such as a gyre, then

$$U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} - 2V\Omega \mathrm{Sin}\phi = -\frac{1}{\rho}\frac{\partial P}{\partial x} \qquad (28)$$

$$U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} + 2U\Omega \mathrm{Sin}\phi = -\frac{1}{\rho}\frac{\partial P}{\partial y} \, . \qquad (29)$$

Some examples of frictional flow are:

Non-accelerated Drift Currents:

Where Coriolis forces are balanced by frictional forces in a homogeneous ocean,

$$2U\Omega \mathrm{Sin}\phi = \varepsilon_z \frac{\partial^2 V}{\partial z^2} \qquad (30)$$

$$-\, 2V\Omega \mathrm{Sin}\phi = \varepsilon_z \frac{\partial^2 U}{\partial z^2} \, . \qquad (31)$$

Solution of these equations gives the so-called "Ekman Spiral" (see Ref. 36) for the motion of the water mass. This type of flow has notable implication for wind-driven drift flow.

Density Flow:

For the case where frictional forces are balanced essentially by a horizontal pressure gradient,

$$\frac{\partial P}{\partial x} = \frac{\partial}{\partial x}\left(\varepsilon_x \frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(\varepsilon_y \frac{\partial U}{\partial y}\right) \qquad (32)$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial x}\left(\varepsilon_x \frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\varepsilon_y \frac{\partial V}{\partial y}\right) \, . \qquad (33)$$

The above examples of frictional and frictionless flows are examples where analytic solutions are possible, but may be difficult to obtain because of complicated boundary conditions.

To completely describe the pollution plume, however, neither inertial terms or eddy diffusion terms can be neglected over the entire range of flow, but certainly some kind of simplification must be made if the problem is to be solved at all. For this reason it has been the course of both theoretical and experimental investigations to consider each flow regime as a separate problem, within which separate simplifications may be warranted.

Simplified Equations for a Forced Vertical Plume

The zone of established flow has received much analytical and experimental attention. In this regime both inertial and viscous forces play an important role in determining the nature of the flow. The simplest case is the axisymmetric vertical plume. Since most of the plume theories center around assumptions fundamental to the solution of the vertical flow problem, the equations for this particular case will be derived from the general equations of motion.

The general assumptions associated with the vertical plume are as follows:

- Steady flow: $\partial U_i/\partial t = 0$
- Flow is axisymmetric: $U_\theta = 0$
- Coriolis effect is neglected: $2e_{ijk}\Omega_j U_k = 0$

- Flow field is assumed hydrostatic throughout: $\frac{\partial P}{\partial z} = -\rho_\infty g$
- Density difference between the plume and surroundings is assumed small compared to the density at any point in the flow field: $|\rho_\infty - \rho| << \rho$
- Plume is fully turbulent
- Eddy transport of momentum and heat is only effective in the lateral direction (normal to jet axis)
- Molecular heat conduction and viscosity ignored.

With the above assumptions and simplifications it is possible to disregard a number of terms in the governing differential equations. Also, for the axisymmetric round jet it is more convenient to consider the general equations in cylindrical coordinates. In simplified form the governing equations become:

Continuity:

$$\frac{1}{r} \frac{\partial (U_r r)}{\partial r} + \frac{\partial W}{\partial z} = 0. \tag{34}$$

Momentum:

Schlicting (1960, ref. 45a) employed an "order of magnitude" analysis incorporating the previous assumptions, which can be used to show that equation (22) reduces to:

$$W \frac{\partial W}{\partial z} + U_r \frac{\partial W}{\partial r} = \left(\frac{\rho_\infty - \rho}{\rho_0}\right) g - \frac{1}{\rho_0 r} \frac{\partial}{\partial r} (r \tau_{rz}). \tag{35}$$

Energy:

In order to deal with specific quantities, a thermal plume will be considered. However, the end result of this derivation will be valid for pollutant plumes in general.

The energy equation (23) reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}\left[\epsilon_{Hr} r \frac{\partial T}{\partial r}\right] = U_r \frac{\partial T}{\partial r} + W \frac{\partial T}{\partial z} . \tag{36}$$

It is sometimes more convenient to consider the energy equation in terms of a temperature disparity,

$$\Theta = T-T_o ,$$

where T_o is a reference temperature, specifically the temperature of the surrounding medium at the jet orifice. The energy equation is then written

$$\frac{1}{r}\frac{\partial}{\partial r} \epsilon_{Hr} r \frac{\partial \Theta}{\partial r} = U_r \frac{\partial \Theta}{\partial r} + W \frac{\partial \Theta}{\partial z} . \tag{37}$$

Equations (35), (36), and (37) may be written in slightly simpler terms by considering the continuity relationship (34). Since

$$\frac{\partial}{\partial z}(W^2) = W \frac{\partial W}{\partial z} + W \frac{\partial W}{\partial z} ,$$

and
$$\frac{\partial}{\partial r}(rU_r W) = rU_r \frac{\partial W}{\partial r} + W \frac{\partial r U_r}{\partial r} ,$$

then equation (35) is written as

$$\frac{\partial W^2}{\partial z} - W \frac{\partial W}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(rU_r W) - \frac{W}{r}\frac{\partial r U_r}{\partial r}$$

$$= \left(\frac{\rho_\infty - \rho}{\rho_o}\right) g - \frac{1}{\rho_o r}\frac{\partial}{\partial r}(r\tau_{rz}). \tag{38}$$

By multiplying equation (34) by W and substituting the result into equation (38), equation (35) is simplified to

$$\frac{\partial}{\partial z}(W^2) + \frac{1}{r}\frac{\partial}{\partial r}(rU_rW) = \left(\frac{\rho_\infty - \rho}{\rho_0}\right)g - \frac{1}{\rho_0 r}\frac{\partial}{\partial r}(r\tau_{rz}). \qquad (39)$$

The same procedure may be applied to the energy equation (36) when the continuity equation is multiplied by T. In this case, equation (36) is simplified to:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[\epsilon_{Hr} r \frac{\partial T}{\partial r}\right] = \frac{1}{r}\frac{\partial}{\partial r}(rU_rT) + \frac{\partial}{\partial z}(WT), \qquad (40)$$

or in terms of a temperature disparity Θ,

$$\frac{1}{r}\frac{\partial}{\partial r}\left[\epsilon_{Hr} r \frac{\partial \Theta}{\partial r}\right] = \frac{1}{r}\frac{\partial}{\partial r}(rU_r\Theta) + \frac{\partial}{\partial z}(W\Theta). \qquad (41)$$

By using the definition of volumetric expansivity (equation 4) a relationship between ρ and T (or Θ) may be developed. By definition

$$\beta = -\frac{1}{\rho}\frac{\partial \rho}{\partial T}, \qquad (4)$$

and to a first order approximation

$$\Delta T = -\frac{1}{\rho_0\beta}\Delta\rho; \qquad (42)$$

so that

$$T - T_0 = \Theta = \frac{1}{\rho_0\beta}(\rho_0 - \rho). \qquad (43)$$

Then equation (43) may be written as

$$\frac{1}{r}\frac{\partial}{\partial r}\left[\epsilon_{Hr} r \frac{\partial(\rho_o-\rho)}{\partial r}\right] = \frac{1}{r}\frac{\partial}{\partial r}\left[rU_r(\rho_o-\rho)\right] + \frac{\partial}{\partial z}\left[W(\rho_o-\rho)\right] \qquad (44)$$

with $\rho_o\beta$ = constant.

Thus, equations (34), (39), and (44) are one form of the set of differential equations which must be solved to establish the flow behavior of a vertical plume.

INITIAL DILUTION OF THE POLLUTION FIELD -- DEVELOPMENT OF WORKING MODELS

In the previous section the fundamental equations governing the motion and dilution of a marine pollution field were developed. It is the purpose of this section to illustrate how analytical and experimental research has proceeded, using these equations, in the development of working mathematical models which describe the fate of a plume from its formation at the orifice to its equilibrium level in the ambient field. But it is not the purpose here to provide a detailed literature review of this subject. Excellent reviews of this material may be found in Hinze (21) for momentum jets and in Baumgartner (6) or Cederwall (11) for buoyant jets.

Research on jet flow has shown that the zone of initial dilution consists of three flow regimes, which are:

- a potential core region, followed by
- a transitional flow, leading to
- the zone of established flow.

In the first two zones, it has been illustrated analytically that similarity solutions* are not possible. On the other hand, once the flow becomes fully established, similarity approximations

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* The premise on which similarity solutions for jet flow are based is that velocity profiles are geometrically similar at all axial locations, differing only by a magnification factor; that is,

$$U(r,z) = f[z \cdot F(r)].$$

For a plume, $U(r,z) = U(z)f(r/b(z))$.

are valid. The transitional zone is usually very short and is generally ignored in model analysis (reference 21) - as will be done here. In the literature, the flow regime between the orifice and the point where the flow becomes established is called "the zone of flow establishment," and the distance required for flow establishment is denoted by Z_e.

Only the mainstream of research on this subject will be presented here - with major emphasis on the zone of established flow, since it is much more important. The discussion on established flow is presented in some detail so that a clear understanding of underlying principles and assumptions is at hand.

Zone of Flow Establishment

As far as the overall plume dispersion is concerned, the zone of flow establishment is of less interest (because it is a comparatively short distance) except as it enters into established flow solutions for integration limits or boundary conditions. Thus, in this light, it is important to know the length for flow establishment, Z_e.

It is usual to assume that the polluted fluid enters the local environment from an orifice as a plug flow (velocity is uniform over the diameter). As the flow penetrates the surrounding fluid, there is a zone of intense shear where eddy mixing causes the flow of pollutant at the edge of the plume to decelerate and the adjacent surrounding fluid to be accelerated. This zone of turbulent mixing

(Figure 9) spreads inward toward the center of the plume and outward into the surrounds as distance from the orifice is increased.

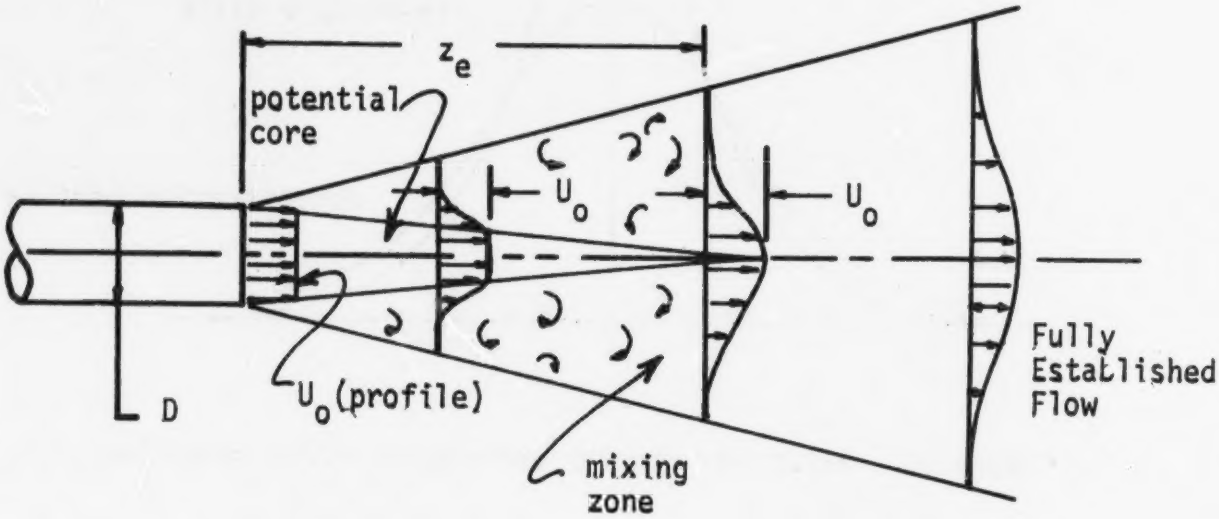


Figure 9. Flow establishment.

The region where the plume flow is undisturbed by the lateral eddy diffusion is usually called the "potential core." Once the lateral spread of eddy diffusion has penetrated to the centerline, the flow is considered to be fully established. Distinction must be made between the development of the velocity profile and the temperature or concentration profile. It has been established experimentally that heat and mass diffuse more rapidly than momentum, in free turbulence. That is to say, that the eddy Prandtl and Schmidt numbers are smaller than 1.0 in free turbulence. The net result of this difference is that temperature and concentration profiles develop more rapidly than do velocity profiles.

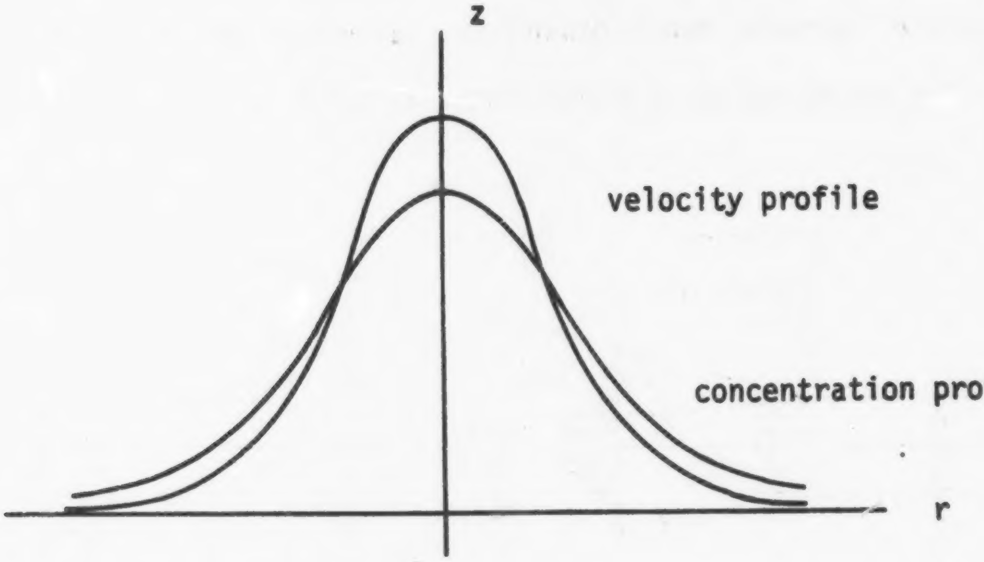


Figure 10. Relative velocity and concentration profiles.

Many experiments have been carried out by various investigators in an effort to establish the length and shape of the potential core for round jets issuing into stagnant fluids. A good review of this work with references is given by Hinze (21). According to Hinze, there is general agreement in most results. The shape of the potential core is evidently conical, and, according to measurements of Hinze and Van der Hegge Zijnen (22), $Z_e \simeq 6D$ to $8D$ for the velocity profile. The work of Albertson, Dai, Jensen and Rouse (4), which seems to be the most frequently quoted, showed that $Z_e \simeq 6.2D$ for the neutrally-buoyant momentum jet.

The results above are associated with neutrally-buoyant jets. If the issuing fluid has significant upward buoyant force which causes changes of momentum similar in magnitude to the initial momentum, the zone of flow establishment takes on different character. For instance, the potential core velocity increases so that at Z_e, $U_m > U_J$, where U_m is the plume centerline velocity. Abraham (1) presents a method for calculating the length for concentration profile establishment in terms of the initial densimetric Froude number Fr_J, as follows:

$$\frac{1.42}{Fr_J} C_2^3 + C_2^2 - \frac{(1+\mu)^2 K}{8} \approx 0, \tag{45}$$

where μ is the eddy Schmidt or Prandtl number and K is an entrainment constant. The densimetric Froude number is given as

$$Fr_J = U_J^2 / \left(\frac{\rho_o - \rho_J}{\rho_J}\right) Dg, \tag{46}$$

where U_J is the orifice velocity, ρ_o density of surrounding fluid at the level of the jet, g acceleration of gravity, and ρ_J density of the issuing fluid. According to Abraham, the constants should take values: $\mu \doteq .80$ and $K = 77$. Then, once Fr_J is established, the quantity C_2 can be found which relates Z_e and D as

$$C_2 = Z_e/D.$$

For $Fr_J \to \infty$, C_2 approaches about 5.6, which is about 10 percent less than the 6.2 found by Albertson et al. for the velocity flow establishment length of neutrally-buoyant jets. The difference here arises from the difference between the rate of spread of matter (or heat) and momentum, for if both are equal, $\mu = 1$, and for $Fr_J \to \infty$, $C_2 \to 6.2$.

The Zone of Established Flow

This particular flow regime has evidently received the majority of theoretical and experimental attention to this time - and, from a practical standpoint, with good reason. This zone is extremely important, because it establishes the initial character of the pollutant dispersion. For instance, with a proper diffuser design (possible only with a reasonably accurate forced plume model) the pollution field can be made to spread at the surface or at a submerged level in a density stratified field, depending on which is the most propitious for the specified water quality situation. For a waste heat discharge, surface spreading will accomplish a greater rate of temperature reduction. Submerged fields of other wastes reduce unsightliness and also may reduce beach hazards associated with onshore surface currents.

Theoretical analysis of neutrally buoyant jet flow dates back to Tollmien's (51) work in 1926, but evidently Schmidt (47) in 1941 was the first to consider the mechanics of convective plumes, or plumes with no initial momentum, such as convective currents over fires, etc., from both the theoretical and experimental views. Schmidt's work was reported in the German literature and, apparently because of the war, went unnoticed until Rouse et al. (45) had carried out similar work in the early 1950's. Since that time a number of researchers have investigated the established flow regime of a forced plume, both theoretically and/or experimentally. From these studies various mathematical models have been proposed and applied to both oceanic and atmospheric problems. Although the models differ in some ways, the theoretical approaches are somewhat similar, and there is reasonably consistent agreement in results.

Particular Cases for Buoyant Plumes

A set of simplified governing equations for a vertical plume issuing from a round port were set down in Chapter III. Even though a vertical plume is not to be expected from an outfall system issuing to receiving waters which are in lateral motion, or for outfalls issuing the pollutant other than vertically, the ideas involved in the vertical plume analysis are basic to more complicated flow problems. For a buoyant plume, researchers have constructed mathematical models for the particular cases which follow (in order of increasing complexity):

1) flow issuing vertically into a stagnant, homogeneous medium;

2) flow issuing vertically into a stagnant, density-stratified medium;

3) flow issuing inclined to the vertical for the above two cases; and

4) flow issuing vertically into a homogeneous medium with a uniform cross-current.

In addition, some work has been carried out for the above cases with a negatively buoyant pollutant, which will be discussed briefly at the end of this section. No models have been developed specifically for an inclined forced plume in a density-stratified medium with a cross-current.

Vertical Plume - Stratified and Non-stratified Media

The analysis of plume dynamics by Schmidt (47) and Rouse et al. (45) dealt with pure buoyancy or thermal convection. Later research has generalized the flow models to include all three classifications of flow.

Rouse, Yih and Humphreys (45). Rouse et al. reported analytical and experimental results for the dynamic and thermodynamic behavior of a convective plume in 1952. In this study, Rouse considered turbulent convective flow above continuous line and point sources of heat rising in a stagnant, homogeneous medium.

To analyze the dynamic and thermodynamic behavior of the axisymmetric plume, Rouse considered the equations (34), (39), and (44). (For the line source, the rectangular counterparts of these equations were used.) A momentum integral equation was derived by integrating equation (39) from r=0 to $r\to\infty$, which is given by:

$$\frac{d}{dz}\int_0^\infty \rho W^2 r dr = -\int_0^\infty \Delta\gamma r dr. \qquad (46)$$

In equation (46), the quantity $\Delta\gamma$ is the buoyant force and symbolically given as:

$$\Delta\gamma = (\rho - \rho_0)g.$$

If the energy equation (44) is integrated over the same range of r, the result is

$$\frac{d}{dz}\int_0^\infty W\Delta\gamma r dr = 0. \qquad (47)$$

As a third equation, Rouse uses a statement of the conservation of mechanical energy, which is derived by multiplying the momentum equation (39) by W and integrating with respect to r to obtain

$$\frac{d}{dz}\int_0^\infty \rho \frac{W^3}{2}\, rdr = -\int_0^\infty W\Delta\gamma rdr - \int_0^\infty \tau_{rz}\frac{\partial W}{\partial r}\, rdr. \tag{48}$$

Now equations (46), (47) and (48) present three equations with which to solve for the three unknowns W, $\Delta\gamma$, and τ. However, these equations cannot be completely solved unless the functional forms of the unknowns are given beforehand. Rouse assumed functions for these quantities in accordance with ideas of dynamic similarity. In this way the following relationships were established:

1) lateral spread, $\sigma \sim z$
2) W_m (centerline) $\sim z^{-1/3}$
3) $\Delta\gamma_m$ (centerline) $\sim z^{-5/3}$.

Finally, it was established through experimentation that axial velocity and temperature were well represented by

$$W(0,z) = W_m = 4.7\left(\frac{-w}{\rho}\right)^{1/3} z^{-1/3} \tag{49}$$

and

$$\Delta T(0,z) = \Delta T_m = -11[\rho(-w)^2]^{1/3} z^{-5/3}, \tag{50}$$

where w represents the mass flow rate given by

$$w = -gH/C_pT. \tag{51}$$

The subscript m relates to plume centerline condition.

In equation (51), H is the heat input and C_p is specific heat. The experimental portion of this study also revealed two other important findings -- first, that velocity and temperature profiles were essentially similar at all elevations; and second, that these profiles could be closely approximated by the Gaussian distribution curve; that is,

$$W(r,z) = W_m \exp[-K_1 r^2/z^2] \tag{52}$$

and

$$\Delta T(r,z) = \Delta T_m \exp[-K_2 r^2/z^2] \tag{53}$$

where K_1 and K_2 are experimentally determined constants found to be 96 and 71, respectively, for the point source.

Priestley & Ball (41). In 1955 Priestley and Ball reported results for studies on thermal plumes rising vertically in a stratified atmosphere. Assumptions are those that give rise to equations (34), (39), and (41). Equation (39) is multiplied through by W, and the term $(\rho_\infty - \rho)/\rho_o$ is replaced by $(T-T_\infty)/T_o$ (perfect gas law, $\rho = P/RT$). Also, in equation (41), $\Theta = T - T_\infty$, instead of $T - T_o$.

The above-mentioned set of equations are then integrated from $r=0$ to $r \to \infty$ to yield the following:

$$\frac{d}{dz}\int_0^\infty \rho W^2 r\,dr = \int_0^\infty \rho g \left(\frac{T-T_\infty}{T_o}\right) r\,dr, \tag{54}$$

$$\frac{d}{dz}\int_0^\infty \frac{1}{2}\,\rho W^3 r dr = \int_0^\infty \rho g W\left(\frac{T-T_\infty}{T_o}\right) r dr - \int_0^\infty \tau\,\frac{\partial W}{\partial r}\,r dr \qquad (55)$$

$$\frac{d}{dz}\int_0^\infty \rho W(T-T_\infty) r dr - \int_0^\infty \rho W r\,\frac{\partial T_\infty}{\partial z}\,dz. \qquad (56)$$

These equations are identical to those used by Rouse, except for nomenclature and the right-hand side of equation (56). This term represents the effect of density stratification which Rouse does not consider.

Priestley then argues that, based on the findings of Rouse, velocity and temperature profiles are Gaussian (including the case of density stratification) and to a first approximation these profiles (heat and momentum) are identical; that is,

$$\frac{W}{W_m} = \frac{\Delta T}{\Delta T_m} = \exp[-r^2/2R_c] \qquad (57)$$

where the subscript again designates conditions at the plume axis and R is a characteristic lateral scale for the plume. It is also assumed that

$$\frac{\tau}{\frac{1}{2}\,\rho {W_m}^2} = f\left(\frac{r}{R_c}\right) \qquad (58)$$

Applying equations (57) and (58) to the equations (54), (55), and (56) yields the following set of differential equations:

$$\frac{d}{dz}\left(R_c{}^2W_m{}^2\right) = 2R_c{}^2\left[\frac{T_m-T_\infty}{T_o}\right] g \tag{59}$$

$$\frac{d}{dz}\left(R_c{}^2W_m{}^3\right) = 3R^2W_m\left[\frac{T_m-T_\infty}{T_o}\right] g - c_1R_cW_m{}^3 \tag{60}$$

$$\frac{d}{dz}\left[R_c{}^2W_m\left(T_m-T_\infty\right)\right] = 2R_c{}^2W_m\frac{dT_\infty}{dz} . \tag{61}$$

From these equations Priestley established that the constant c_1 (a profile constant or spreading constant) may be determined by

$$\frac{dR_c}{dZ} = c_1$$

which results by combining equations (59) and (60).

For neutral conditions in the atmosphere ($dT_\infty/dz = 0$), the general solutions are:

$$W_m = \left[\frac{3Ag}{2T_oc_1{}^2z} + \frac{B}{z^3}\right]^{1/3} \tag{62}$$

and $$T_m-T_\infty = \frac{A}{c_1{}^2z^2}\left[\frac{3Ag}{2T_oc_1{}^2z} + \frac{B}{z^3}\right]^{-1/3} , \tag{63}$$

where A and B are constants of integration and may be determined from boundary conditions.

For the case where $dT_\infty/dz \neq 0$, apparently no analytical solution exists. However, it is a relatively simple task to carry out numerical solutions with a digital computer.

Priestley and Ball reported experimental results for temperatures measured between 5 and 70 cm above the heat source for the neutral (non-stratified) case. These results did not compare favorably with equation (63) (calculated results were about 75 percent too high); but the measured results did have the same general shaped curve as equation (63). In other words,

$$T_m - T_\infty \sim \frac{A}{c_1{}^2 z^2} \left(\frac{3Ag}{2T_o c_1{}^2 z} + \frac{B}{z^3} \right)^{-1/3} \qquad (64)$$

Measurements transverse to the axis of flow revealed that the temperature profiles were indeed nearly Gaussian.

Morton, Taylor, and Turner (34). Morton, Taylor and Turner also studied the problem of gravitational convection from a continuous point source. Although they employed the same fundamental principles as previous investigators, their development has a slightly different approach, which has been adopted by several other investigators. One difference between this development and that of Priestley and Ball (36) is that here an unknown entrainment velocity must be contended with, whereas Priestley and Ball dealt with an unknown spreading coefficient.

At this point two assumptions are made:

1) The entrainment velocity is proportional to the vertical plume velocity; and

2) that the plume has some defined boundary b; that is, R=b, where b is not the actual radius of the plume, but some characteristic radius.

Morton considers two different velocity profiles (see Figure 11) as noted:

1) the so-called "top hat," or

$$W(r,z) = W_m, \ r \le b,$$

$$\frac{\rho_\infty - \rho(r,z)}{\rho_o} = \frac{\rho_\infty - \rho_m}{\rho_o}, \ r \le b;$$

2) Gaussian, or

$$W(r,z) = W_m \exp[-(\frac{r}{b})^2]$$

$$\frac{\rho_\infty - \rho(r,z)}{\rho_o} = \left(\frac{\rho_\infty - \rho_m}{\rho_o}\right) \exp[-(\frac{r}{b})^2]$$

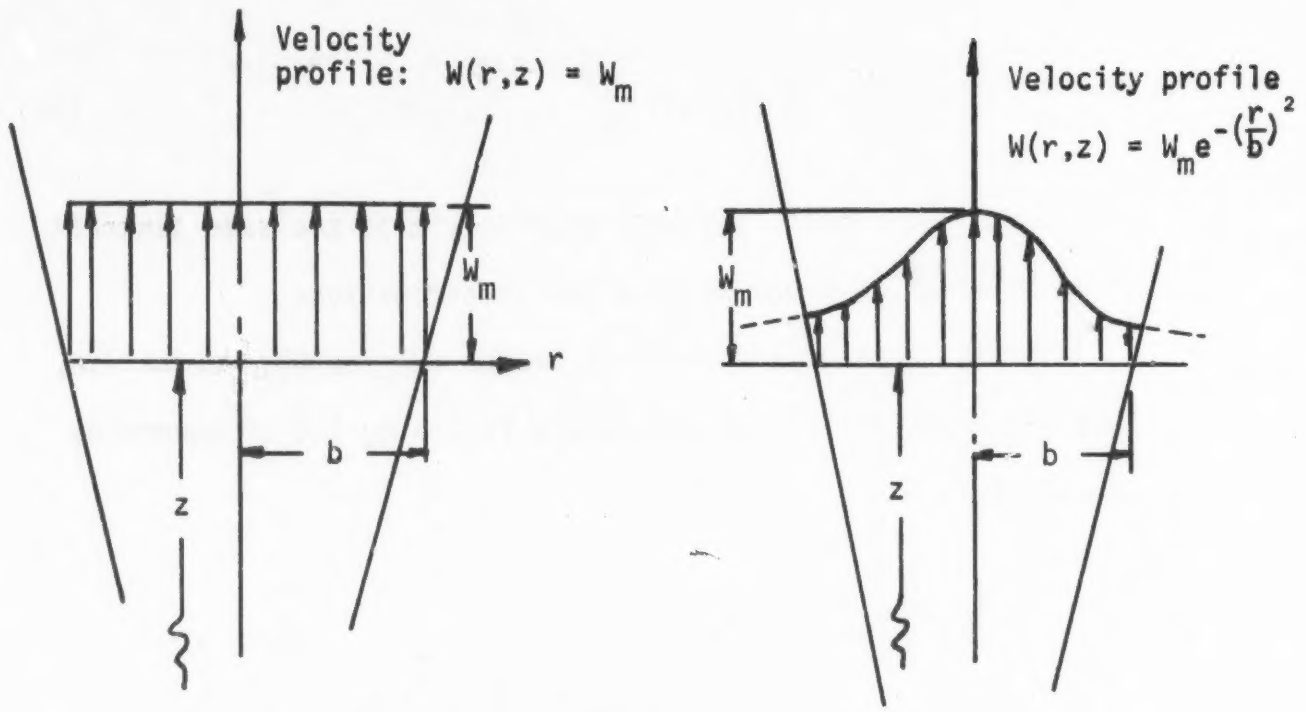
Use of the "top hat" profile and the above two assumptions yields:

$$\frac{d}{dz}(b^2 W_m) = 2b\alpha W_m \tag{65}$$

$$\frac{d}{dz}(b^2 W^2{}_m) = b^2 g \frac{\rho_\infty - \rho_m}{\rho_o} \tag{66}$$

$$\frac{d}{dz}\left\{b^2 W_m g \left(\frac{\rho_\infty - \rho_m}{\rho_o}\right)\right\} = \frac{b^2 W_m g}{\rho_o} \frac{d\rho_\infty}{dz}, \tag{67}$$

as the set of governing equations.


Velocity profile: $W(r,z) = W_m$
W_m
r
b
z
Velocity profile $W(r,z) = W_m e^{-(\frac{r}{b})^2}$
W_m
b
z

"Top hat" profile Gaussian profile

Figure 11. Velocity profile assumed by Morton.

It is assumed that α is constant with respect to changes of W_m or z, which may not be the actual case. Also, α must be determined from experimental or observational data.

For the case of a non-stratified medium $\rho_\infty=\rho_o$ and $d\rho_\infty/dz=0$, so that from (67)

$$b^2 W_m g \left(\frac{\rho_o-\rho_m}{\rho_o}\right) = \text{constant}, \tag{68}$$

and for this case the solutions are:

$$b = \frac{6}{5}\alpha z, \tag{69}$$

$$W_m = \frac{5}{6}\alpha\left(\frac{9}{10}\alpha Q\right)^{1/3} z^{-1/3}, \tag{70}$$

$$g\left(\frac{\rho_o-\rho_m}{\rho_o}\right) = \frac{5Q}{6\alpha}\left(\frac{9}{10}\alpha Q\right)^{-1/3} z^{-5/3} . \qquad (71)$$

Thus, the simple "top hat" profile yields the same general flow behavior as found by previous investigations.

For the case of a stratified medium ($d\rho_\infty/dz \neq 0$), Morton uses the Gaussian profiles to obtain the following set of governing equations:

$$\begin{aligned} &\frac{d}{dz}(b^2W_m) = 2\alpha bW_m \\ &\frac{d}{dz}(b^2W_m^{\,2}) = 2b^2g\left[\frac{\rho_\infty-\rho_m}{\rho_o}\right] \qquad (72) \\ &\frac{d}{dz}\left[b^2W_m g\,\frac{\rho_\infty-\rho_m}{\rho_o}\right] = 2b^2W_m\,\frac{g}{\rho_o}\,\frac{d\rho_\infty}{dz} . \end{aligned}$$

Note that a factor 2 appears on the right-hand side of the last two equations, the only difference between this set and (65), (66), and (67). These equations were rewritten with normalized variables and solved numerically. For a linearly-stratified medium, the dimensionless z is found to achieve a maximum value of 2.8. This is related to the actual maximum height of rise, through use of a scale factor proportional to the fourth root of the initial buoyancy flux and inversely proportional to the 3/8 power of the uniform gradient, i.e.

$$z_{max} \sim \frac{[Q_o g(\rho_o-\rho_j)]^{1/4}}{\left(\frac{\Delta\rho_\infty}{\Delta z}\right)^{3/8}}$$

Since the equations were developed for a point source, the height z_{max} is measured from a virtual source (see Figure 12) which lies some distance below the real source (which has finite diameter). The virtual source for their experiments was found to be

$$z_s = \frac{D}{.48} ,$$

where z_s is the distance the virtual source lies below the real source and D is the real source diameter.

From the same experiments it was also found that

$$\alpha = .093,$$

the proportionality constant between the vertical velocity at the plume centerline and the lateral entrainment velocity.

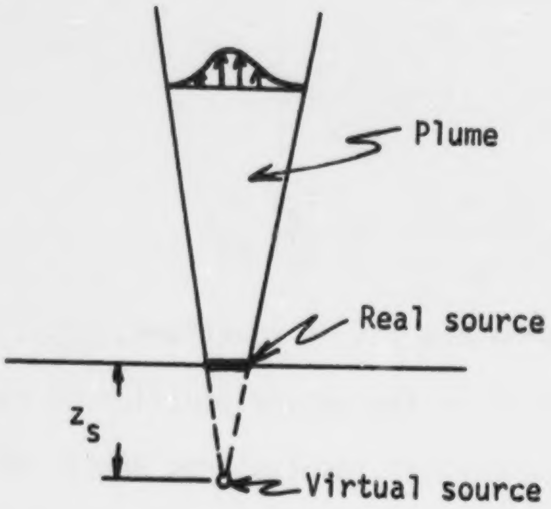


Figure 12. Virtual source.

Morton (33). Morton extended this work in two papers published in 1959. In this work he assumed that the characteristic lengths for the velocity and temperature (or buoyancy) profiles are different to account for the difference between the eddy transport of heat and momentum. He also extended the 1956 work to include a flux of mass and momentum from the source, whereas the 1956 work considered only a case of pure buoyancy such as a "thermal."

To include the effect of a different rate of spread between heat and momentum, he assumes the characteristic length for the velocity profile to be b as before, but the characteristic length of heat (or buoyancy) to be λb. Under these assumptions, the governing equations:

$$\begin{aligned} &\frac{d}{dz}[b^2 W_m] = 2\alpha b W_m \\ &\frac{d}{dz}[b^2 W_m{}^2] = \lambda^2 b^2 g\left(\frac{\rho_\infty - \rho_m}{\rho_o}\right) \qquad (74) \\ &\frac{d}{dz}\left[b^2 W_m g\left(\frac{\rho_\infty - \rho_m}{\rho_o}\right)\right] = b^2 \frac{W_m g}{\rho_o}\frac{d\rho_\infty}{dz} . \end{aligned}$$

The only change from the set of equations, (65), (66), and (67), is the appearance of λ^2 in the second equation of set (74).

Using the experimental results from Rouse et al. for a point source, Morton ascertained that λ=1.16, and from his previous experimental results, α=.082, if Gaussian profiles are assumed. In the case of "top hat" profiles, α is multiplied by $\sqrt{2}$ to obtain α=.116 and $\lambda = \sqrt{\frac{1+1.16}{2}} = 1.08$.

The assumptions in Morton's model that entrainment occurs through the rise of the plume to z_{max} and that the similarity of profiles is likewise maintained implies that no fluid leaves the plume until z_{max}, at which point it is not accounted for. Schematically, this implication is superimposed on the observed pattern in Figure 13.

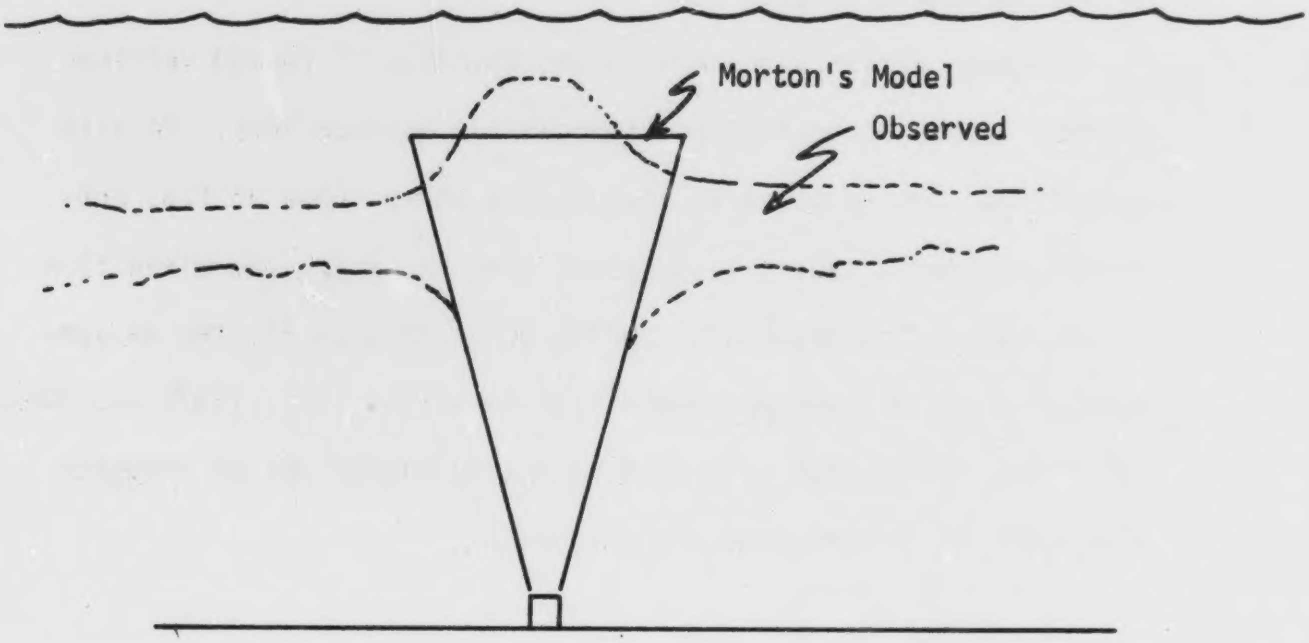


Figure 13. Comparison of Morton's model to representative observation.

The consequence of this is that the model underestimates the penetration of the plume (z_{max}) if one accepts Morton's entrainment coefficient, or one must find a lower value for the entrainment coefficient to match the heights. In any event, there is no information provided for the thickness or location of the spreading field, which is felt to be more important than z_{max}. Morton et al. (34), however, do point out that the plume should begin to spread horizontally at the level where $\Delta\rho$ is first diminished to zero.

Abraham (1). Abraham sought to overcome part of this objection by allowing entrainment to become negative above the equilibrium level, thus accounting in an artificial manner for transfer of mass to the horizontal layer.

Abraham considers the motion and dilution of forced vertical plumes in both stratified and homogeneous surroundings. He also generalizes the solution to include the three types of flow mentioned earlier: neutrally buoyant, pure buoyancy, and mixed flow.

To derive the equations considered by Abraham for the axisymmetric round jet, one may begin with equations (34), (39), and (44). Following integration, one obtains the following set of integral equations for a homogeneous environment:

$$\begin{aligned} &\frac{d}{dz}\int_0^{R_b} 2\pi W r dr = -2\pi R_b U_r(R_b) \\ &\frac{d}{dz}\int_0^{R_b} \rho 2\pi W^2 r dr = \int_0^{R_b} 2\pi(\rho_\infty-\rho) g r dr \qquad (75) \\ &\frac{d}{dz}\int_0^{R_b} 2\pi W(\rho_o-\rho) r dr = 0. \end{aligned}$$

Abraham also assumes that the velocity and buoyancy profiles are Gaussian, but differ in the dispersion scale to account for differences of the lateral eddy transport of momentum and matter (or heat). These assumed distributions have the form:

$$W(r,z) = W_m \exp[-K(\frac{r}{z})^2] \tag{76}$$

$$C(r,z) = C_m \exp[-K\mu(\frac{r}{z})^2] \tag{77}$$

where K and μ are experimentally determined entrainment coefficients for momentum and matter, respectively, and C_m is concentration defined as

$$C_m = \frac{\rho_o-\rho(0,z)}{\rho_o-\rho_J} = \frac{\rho_o-\rho_m}{\rho_o-\rho_J} . \tag{78}$$

In the above equation, ρ_J is the effluent density as it leaves the orifice. Note that the above definition of C_m is not a valid definition of concentration for pollutants in stratified ambient media. In the general case the numerator must contain ρ_∞ instead of ρ_o.

For the purpose of integration, Abraham permits $R_b \to \infty$, and the equation set (75) is written as:

$$\frac{d}{dz}\int_0^\infty 2\pi Wrdr = Q' \tag{79}$$

$$\frac{d}{dz}\int_0^\infty 2\pi\rho_J W^2 rdr = \int_0^\infty 2\pi(\rho_o-\rho)grdr \tag{80}$$

$$\frac{d}{dz}\int_0^\infty 2\pi W(\rho_0-\rho)rdr = 0. \tag{81}$$

In the continuity equation (79) above, Abraham does not assume a relationship between W and U_r as did Morton et al. (34), but simply insists that continuity is satisfied by some unknown inflow, Q', to the plume. Also in the second equation Abraham uses ρ_J instead of ρ and in the second and third equations ρ_∞ has been replaced by ρ_0 since $\rho_0=\rho_\infty$ for a nonstratified medium.

If equation (80) is multiplied by dz and the result integrated from z_e to z (z_e is the point where the flow becomes fully established - see discussion on zone of flow establishment), the result is:

$$2\pi\rho_J\int_0^\infty W^2rdr = M_e + g\int_{z_e}^{z} dz \int_0^\infty 2\pi(\rho_0-\rho)rdr, \tag{82}$$

where M_e is the jet momentum at $z = z_e$ or the left-hand side of the equation evaluated at $z = z_e$. The following expression for the plume centerline momentum as a function of z is obtained when (76), (77), and (78) are inserted into (82) and integrated:

$$\frac{\pi}{2K} W_m^2 \rho_J z^2 = M_e + g\pi \int_{z_e}^{z} \frac{(\rho_0-\rho_J)}{\mu K} C_m z^2 dz. \tag{83}$$

When equation (81) is divided by $(\rho_0-\rho_J)$, (76) and (77) are used to replace W and the quantity $(\rho_0-\rho)/(\rho_0-\rho_J)$, and the result integrated from z_e to z:

$$\frac{\pi}{K(1+\mu)} C_m W_m z^2 = \frac{\pi D^2}{4} W_J C_J \tag{84}$$

is obtained, where C_J and W_J are plume conditions at the orifice and D is the orifice diameter.

By differentiating (83) with respect to z and using (84) to eliminate C_m, the following differential equation is obtained.

$$(W_m{}^2 z^2)^{1/2} \frac{d(W_m{}^2 z^2)}{dz} = \frac{K(1+\mu)}{4\mu} g \left[\frac{\rho_o - \rho_J}{\rho_J}\right] D^2 W_J C_J 2z. \tag{85}$$

Equation (85) can then be integrated to obtain

$$\frac{W_m}{W_J} = \frac{D}{z}\left\{\left[\frac{K}{2}\frac{M_e}{M_J}\right]^{3/2} + \frac{3}{8}\frac{1}{Fr_J}\left(\frac{1+\mu}{\mu}\right)K\left[\frac{z^2}{D^2} - \frac{z_e{}^2}{D^2}\right]\right\}^{1/3} . \tag{86}$$

In equation (86),

$$M_J = (\pi/4) D^2 W_J{}^2 \rho_J$$

and $$Fr_J = W_J{}^2 / \left[\frac{\rho_o - \rho_J}{\rho_J} gD\right],$$

and the lower limit of integration established by (83) is given as

$$W_m{}^2 z_e{}^2 = \frac{2K}{\pi\rho_J} M_e ,$$

at $z = z_e$. Then from equation (84),

$$\frac{C_m}{C_J} = \frac{K(1+\mu)}{4} \left(\frac{D}{z}\right)^2 \frac{W_J}{W_m} . \tag{87}$$

Hence, the dilution of a vertical plume

$$S_m(r,z) = \frac{1}{C_m(r,z)} \tag{88}$$

is found from equations (86) and (87). As mentioned previously in the discussion on the zone of flow establishment, the quantity z_e is found by the equation:

$$\frac{1.42}{Fr_J}\left(\frac{z_e}{D}\right)^3 + \left(\frac{z_e}{D}\right)^2 - \frac{(1+\mu)^2 K}{8} = 0, \tag{89}$$

where the values of k and μ are 77 and .80, respectively, for equation (89).

Abraham also presents solutions for the limiting cases where either inertial forces dominate the flow behavior ($FR_J \to \infty$) or where buoyant forces dominate ($FR_J \to 0$). For the uniform medium the following limiting solutions are available:

- Inertial dominated flow: $Fr_J \to \infty$

$$\frac{W_m}{W_J} = \left(\frac{K}{2}\right)^{1/2} \frac{D}{z}, \tag{90}$$

$$\frac{C_m}{C_j} = \left(\frac{1+\mu}{2}\right)\left(\frac{K}{2}\right)^{1/2} \frac{D}{z} \tag{91}$$

in which case K=77 and μ=.80.

Buoyancy dominated flow: $FR_J \to 0$

$$\frac{W_m}{W_J} = \frac{D}{z}\left[\frac{3}{8}\,\frac{1}{Fr_J}\left(\frac{1+\mu}{\mu}\right)K\left(\frac{z^2}{D^2} - \frac{z_e^{\,2}}{D^2}\right)\right]^{1/3} \qquad (92)$$

$$\frac{C_m}{C_J} = K\,\frac{(1+\mu)}{4}\left(\frac{D}{z}\right)^2\frac{W_J}{W_m}\,, \qquad (87)$$

in which case K=92 and μ=.74.

Note that the values of K and μ differ depending on whether the flow is inertial or buoyancy-dominated. Normally the flow in the plume would be mixed (with respect to inertial and buoyancy domination) so that over the established flow regime $77 \le K \le 92$ and $.74 \le \mu \le .80$. Abraham discussed which values of K and μ should be used for calculations in reference (1). Abraham's solutions (86) and (87) for the nonstratified case are identical to those given by Priestley and Ball (41) except, as Abraham says, "The physical parameters are more clearly defined."

Abraham also discusses flow in a stratified medium. Since an analytical solution cannot be obtained for this case, the approach used by Abraham is to divide the medium depth into a number of increments, each with uniform but different density, and then systematically apply equations (86) and (87) for the solution. If the subscript n refers to an increment identification number, in sequential order, the equations for numerical solution are:

$$\frac{W_m}{W_{m,n-1}} = \frac{z_{n-1}}{z}\left[K_1 + K_2\,\frac{z^2}{z^2_{n-1}}\right]^{1/3} \qquad (93)$$

$$\frac{C_m}{C_{m,n-1}} = \frac{z_{n-1}}{z}\left[K_1 + K_2\,\frac{z^2}{z^2_{n-1}}\right]^{-1/3} \qquad (94)$$

where

$$K_1 = 1 - \frac{3}{2}\,\frac{1}{\mu F_{n-1}},$$

$$K_2 = \frac{3}{2}\,\frac{1}{\mu F_{n-1}},$$

$$F_{n-1} = \frac{(W_{m,n-1})^2}{\left[\dfrac{\rho_n - \rho_{m,n-1}}{\rho_j}\right] g z_{n-1}},$$

and $C_m = \dfrac{\rho - \rho_n}{\rho_{m,n-1} - \rho_n}$.

Inclined and Horizontal Plumes - Stagnant Environment

In more recent outfall construction it has been the practice to orient the diffuser ports so that the waste material is issued horizontally into the receiving water (Figure 14). Both theory and practice have shown that the horizontal plume is more efficient in diluting the pollutant with respect to the final height the pollution field achieves. The analyses offered thus far for the horizontal plume differ little from vertical plume analysis, except that a horizontal component of momentum is considered.

Abraham (1). Abraham also presents an analysis for horizontal round plumes issuing into stagnant, nonstratified media.

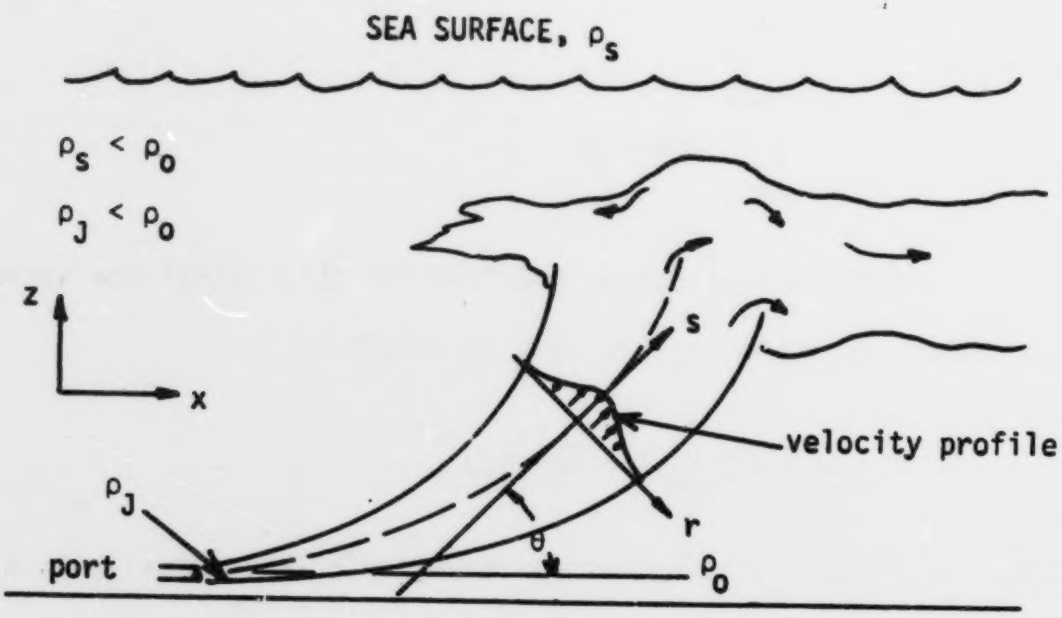

SEA SURFACE, ρ_s
$\rho_s < \rho_o$
$\rho_J < \rho_o$
z
x
s
velocity profile
ρ_J
r
θ
port
ρ_o

Figure 14. Horizontal plume issuing into stagnant, density stratified medium.

For the horizontal plume, the natural coordinates are s and r (Figure 14), where s is the distance along the plume axis and r always lies in a plane normal to this direction. Although the velocity and buoyancy profiles are certainly not Gaussian, they are assumed as such to simplify the analysis. These profiles are represented by:

$$U(r,s) = U_m \exp[-K(\frac{r}{s})^2] \tag{95}$$

and

$$C(r,s) = C_m \exp[-K\mu(\frac{r}{s})^2]. \tag{96}$$

The z-x coordinates for the flow are given by:

$$\frac{z}{D} = \int_0^{s/D} \mathrm{Sin}\theta d(\tfrac{s}{D}), \tag{97}$$

$$\frac{x}{D} = \int_0^{s/D} \mathrm{Cos}\theta d(\tfrac{s}{D}). \tag{98}$$

Equation (80) can be modified for this coordinate system simply by replacing z with s to obtain:

$$\frac{\pi}{K(1+\mu)} C_m U_m s^2 = \frac{\pi D^2}{4} C_J U_J. \tag{99}$$

However, in the momentum equation both z-directional and x-directional momentum must be considered. For the z-direction, the equation analogous to (82) is

$$\mathrm{Sin}\theta \int_0^\infty 2\pi \rho_J U^2 r dr = M_e \mathrm{Sin}\theta_e + g \int_{s_e}^{s} ds \int_0^\infty 2\pi(\rho_o - \rho) r dr, \tag{100}$$

and for the x-direction (horizontal):

$$\mathrm{Cos}\theta \int_0^\infty \rho_J U^2 r dr = M_J, \tag{101}$$

where M_J is the momentum at the orifice.

Equations (100) and (101) are integrated using profiles (95) and (96) and added vectorially to obtain

$$\left\{ \left(\frac{\pi}{2K} U_m{}^2 \rho_J s^2\right)^2 - \left(\frac{\pi D^2}{4} U_J{}^2 \rho_o\right)^2 \right\}^{1/2} = M_e \mathrm{Sin}\theta_e + g\pi \int_{s_e}^{s} \frac{\rho_o - \rho_j}{\mu K} C_m s^2 ds. \tag{102}$$

With use of equation (99), equation (102) may be reduced to the differential form

$$\frac{U_m}{U_J}\frac{s}{D}\, d\left\{\left[\frac{U_m}{U_J}\frac{s}{D}\right]^4 \frac{1}{K^2} - \frac{1}{4}\right\}^{1/2} = \frac{1+\mu}{2\mu}\frac{1}{Fr_J}\frac{s}{D}\, d\left(\frac{s}{D}\right). \tag{103}$$

Equation (103) is then used to solve for U_m/U_J as a function of s/D. Equation (99) is used to evaluate C_m/C_J, equation (101) to find θ and eventually (97) and (98) to find z/D and x/D. To evaluate the entrainment coefficients, Abraham uses the following relationships:

$$K = -304\left(\frac{\theta}{\pi}\right)^3 + 228\left(\frac{\theta}{\pi}\right)^2 + 77 \tag{104}$$

$$\mu = .96\left(\frac{\theta}{\pi}\right)^3 - .72\left(\frac{\theta}{\pi}\right)^2 + .80. \tag{105}$$

Obviously, the solution for the horizontal plume problem must be carried out numerically. Also, there is some question concerning the basis of equations (104) and (105) (Fan (14)).

Fan (14). Work carried out by Fan and reported in 1967 represents valuable extensions to established jet and plume theory. Part of this work is concerned with plumes issuing horizontally (and at arbitrary angles above the horizontal) into a stagnant, linearly density-stratified medium. Fan's work is based on the Morton et al. (34) theoretical approach, along with the Albertson et al. (4) experimental work to define the length for flow establishment.

In Figure 15 the z',x' coordinates are referred to the actual source, and the z,x coordinates are referred to the point where the flow becomes fully established with respect to the velocity profile, or 6.2D from the z',x' origin at an angle θ_0.

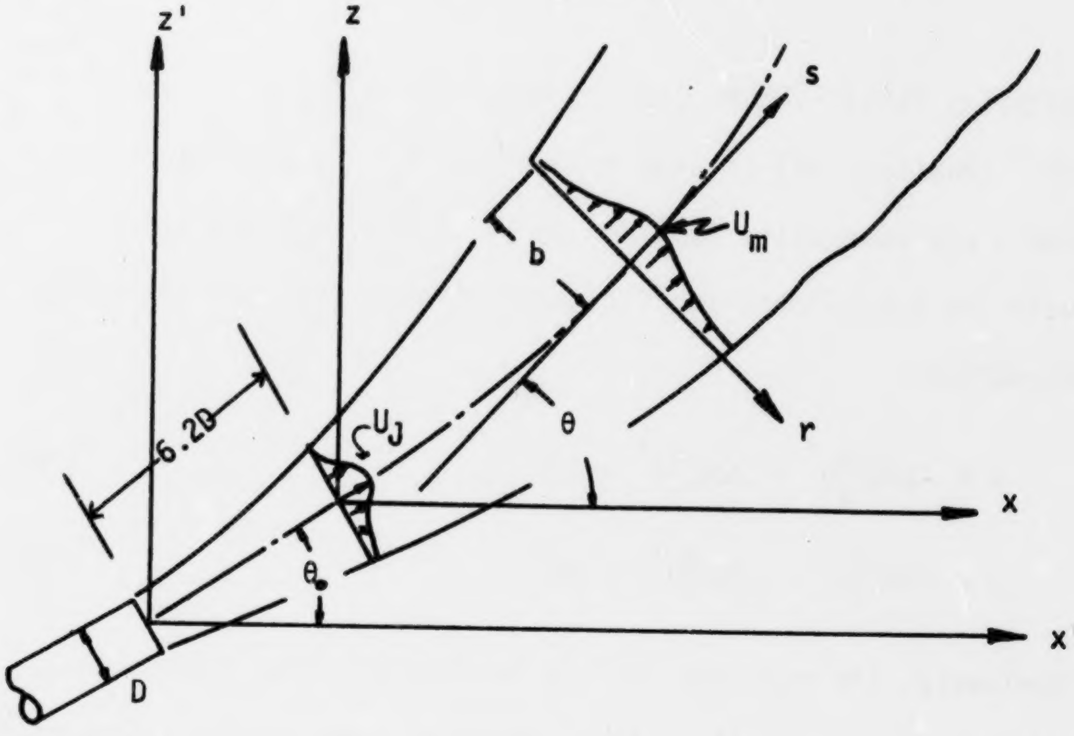


Figure 15. Plume issuing at an angle into a stagnant medium.

Equations are again needed for continuity, z and x momentum, buoyancy, and geometry. These expressions are written in terms of coordinates along the jet axis, s, and a normal to this direction, r. Velocity and buoyancy profiles are assumed Gaussian in the region of established flow:

$$U(r,s) = U_m(s)\exp[-r^2/b^2]$$

$$\rho_\infty - \rho = [\rho_\infty - \rho_m(s)]\exp[-r^2/(\lambda^2 b^2)],$$

where b is a characteristic width.

The governing equations are as follows:

Continuity: $$\frac{d}{ds}(b^2 U_m) = 2bU_m\alpha \quad (106)$$

z-momentum: $$\frac{d}{ds}\left(\frac{b^2 U_m^2}{2}\operatorname{Sin}\theta\right) = g\lambda^2 b^2 \frac{\rho_\infty - \rho_m}{\rho_o} \quad (107)$$

x-momentum: $$\frac{d}{ds}\left(\frac{b^2 U_m^2}{2}\operatorname{Cos}\theta\right) = 0 \quad (108)$$

Buoyancy: $$\frac{d}{ds}(U_m b^2[\rho_\infty - \rho_m]) = \frac{1+\lambda^2}{\lambda^2} b^2 U_m \frac{d\rho_\infty}{ds} \quad (109)$$

z-coordinate: $$\frac{dz}{ds} = \operatorname{Sin}\theta \quad (110)$$

x-coordinate: $$\frac{dx}{ds} = \operatorname{Cos}\theta \quad (111)$$

Concentration: $$C_m U_m b^2 = \text{Const.} = U_J C_o b_o^2 \quad (112)$$

To obtain the solutions in terms of U_m, b, θ, $\rho_\infty - \rho_m$, x, and z, the above six differential equations must be solved simultaneously. The initial conditions are $U_m(0)=U_J$, $b(0)=b_o(b_o=D/\sqrt{2})$, $\rho_m(0)=\rho_J$, $\theta(0)=\theta_o$, and x=z=0 at s=0. Fan normalized these equations in much the same fashion as did Morton et al. (34) and worked with dimensionless parameters. Solutions were carried out numerically by digital computer (see reference 14a).

Cederwall (11). Based on equations developed by Bosanquet et al. (8) for plumes issuing into fluids of other density at various inclined angles, Cederwall proposed the following set of equations for horizontal plumes in homogeneous receiving waters:

$$S_m = \frac{C_J}{C_m} = .54\ Fr_J^{1/2}\left[\frac{z}{DFr_J^{1/2}}\right]^{7/16} \quad ; \ z/D < .5\ Fr_J^{1/2} \tag{113}$$

$$S_m = \frac{C_J}{C_m} = .54\ Fr_J^{1/2}\left[.38\ \frac{z}{DFr_J^{1/2}} + .66\right]^{5/3} ; \ z/D \geq .5\ Fr_J^{1/2} \tag{114}$$

$$\frac{U_J}{U_m} = .40\ Fr_J^{1/2}\left[\frac{z}{DFr_J^{1/2}}\right]^{2/11} . \tag{115}$$

These equations are for conditions along the jet axis, with the zone of flow establishment being neglected. Cederwall comments that Abraham's model can be written in similar form if the zone of flow establishment is neglected.

For the case where the receiving water is stratified, Cederwall suggests a step-by-step numerical approach using the above set of equations. For application, Reference 11 should be consulted.

Trent, Baumgartner and Byram (52). Theoretical work on inclined jets issuing into arbitrarily stratified media is also being carried on at Corvallis by Trent et al. This work follows more along the work of Abraham than that of Morton et al. It is felt that the introduction of the characteristic length, b, used by Morton (and later by Fan) is an unneeded complication.

Refer to Figure 15 and consider the velocity and density disparity profiles to be given by:

$$U(r,s) = U_m(s)\exp[-K(r/s)^2] \tag{116}$$

$$\frac{\rho_\infty - \rho(r,s)}{\rho_o - \rho_J} = \frac{\rho_\infty - \rho_m(s)}{\rho_o - \rho_J}\exp[-K\mu(r/s)^2] \tag{117}$$

where K and μ are possible functions of s. Let

$$E = U_m{}^3,$$

$$\Delta_m = \frac{\rho_\infty - \rho_m(s)}{\rho_o - \rho_J},$$

$$R = U_m \Delta_m.$$

The governing equations become, after simplification:

Axial velocity:

$$\frac{dE}{ds} + \frac{3E}{s} - \frac{E}{2K}\frac{dK}{ds} = \frac{3g}{\mu}\left(\frac{\rho_o - \rho_J}{\rho_J}\right) R \operatorname{Sin}\theta \tag{118}$$

Density disparity:

$$\frac{dR}{ds} + \frac{2R}{s} - \frac{R}{K(1+\mu)}\frac{d[K(1+\mu)]}{ds} = \frac{1+\mu}{\rho_o - \rho_J} E^{1/3} \frac{d\rho_\infty}{ds} \tag{119}$$

Angle:

$$\theta = \operatorname{Cos}^{-1}\left(\frac{E_e}{E}\right)^{2/3}\left(\frac{S_e}{s}\right)^2 \operatorname{Cos}\theta_0 \tag{120}$$

Geometry:

$$x = x_e + \int_{s_e}^{s} \operatorname{Cos}\theta\, d\xi \tag{121}$$

$$z = z_e + \int_{s_e}^{s} \operatorname{Sin}\theta\, d\xi \tag{122}$$

Continuity of pollutant:

$$C_m = \left(\frac{E_e}{E}\right)^{1/3} \left(\frac{S_e}{s}\right)^2 C_o \tag{123}$$

where C_m is any convenient measure of centerline concentration. It should be pointed out here that in the case of a hot water plume having the same salinity as the ambient, Δ_m is the measure of pollutant concentration and equation (123) is unnecessary.

In order to evaluate equations (118) and (119), it is necessary to evaluate

$$\frac{1}{2K}\frac{dK}{ds}$$

and $$\frac{1}{K(1+\mu)}\frac{d[K(1+\mu)]}{ds} .$$

These quantities are functions of the as-yet-undefined flow field. And, even if the flow dynamics were known, just how K and μ vary is open to question. We do have, however, Abraham's limiting values for small and large Fr_J (see page 57). In this work we assume that K and μ change slowly along s, at least over the range of s where a similarity solution is valid. Then terms involving derivatives of K and μ are small compared to other terms of the equations and as a first approximation may be ignored.*

---

* Once the approximate flow dynamics have been established, one could, at least in principal, use this information to estimate derivatives of K and μ and thus improve the solution.

Equations (118) and (119) become:

$$\frac{dE}{ds} + \frac{3E}{s} = \frac{3g}{\mu}\left[\frac{\rho_o-\rho_J}{\rho_J}\right]R\text{Sin}\theta \tag{124}$$

$$\frac{dR}{ds} + \frac{2R}{s} = \frac{1+\mu}{\rho_o-\rho_J}\,E^{1/3}\,\frac{d\rho_\infty}{ds}\,. \tag{125}$$

The quantity R could be eliminated from (124) by using (125), which would yield a second order equation in E. However, the equations will eventually be solved numerically, so that two first-order equations turn out to be the more convenient form.

The governing equation may be non-dimensionalized by defining the following parameters:

$$E^* = E/E_J = U_m{}^3/U_J{}^3$$

$$s^* = s/D,\ x^* = x/D,\ z^* = z/D$$

$$R^* = R/R_J = U_m\Delta_m/U_J\Delta_J$$

$$\rho_\infty{}^* = \rho_\infty/(\rho_o-\rho_J).$$

Hence, the equation set in dimensionless form becomes:

$$\frac{dE^*}{ds^*} + \frac{3E^*}{s^*} = \frac{3}{\mu Fr_J}R^*\text{Sin}\theta \tag{126}$$

$$\frac{dR^*}{ds^*} + \frac{2R^*}{s^*} = (1+\mu)E^{*1/3}\,\frac{d\rho_\infty{}^*}{ds^*}\,, \tag{127}$$

along with appropriate dimensionless forms for equations (120), (121), (122), and (123) if needed. Initial conditions are:

$$\text{at } s^* = s_e^*: \quad E^* = E_e^* = 1$$

$$R^* = R_e^* = 1$$

$$\theta = \theta_o$$

$$C_m^* = C_e^* = 1.$$

Here s_e^* is 5.6, based upon concentration profile establishment for $Fr_J \geq 100$.

From equations (126) and (127), the obvious simplifications are:

- Vertical buoyant plume: $\theta = \text{Const} = 90°$
- Homogeneous medium: $d\rho_\infty^*/ds^* = 0$.

These equations also reveal two similarity parameters (aside from θ):

- $$Fr_J = \frac{U_J^2}{\left(\frac{\rho_o - \rho_J}{\rho_J}\right) Dg}$$
- $$\frac{d\rho_\infty^*}{ds^*} \text{ ; or } \frac{d\rho_\infty^*}{dz^*} = \frac{d\left(\frac{\rho_\infty}{\rho_o - \rho_J}\right)}{d(z/D)} .$$

For linear stratification:

- $$\frac{d\rho_\infty^*}{dz^*} = -\left|\frac{\Delta\rho_\infty}{L}\right|\left(\frac{D}{\Delta\rho_J}\right) = -G$$

where G is the stratification parameter, and $\Delta\rho_J = \rho_o - \rho_J$.

Numerous calculations have been carried out based on various values of G and Fr_J. The dimensionless height of z^*_{max} and corresponding downstream distance x^*_{max} are plotted in Figure 16 for parametized values Fr_J and G, where $\theta_0=0°$. Figure 17 illustrates z^*_{max} for a vertical jet as a function of Fr_J for various values of G. These results are in excellent agreement with Fan's experimental results, and may be used as a guide for ocean outfall diffuser designs.

The information in Figure 17 may be approximated by the equation:

$$z^*_{max} = \frac{11}{3} Fr_J^{1/8} G^{-3/8} . \tag{129}$$

For values of $G \geq 2 \times 10^{-3}$ this equation begins to lose accuracy, as illustrated by the dashed line in Figure 17.

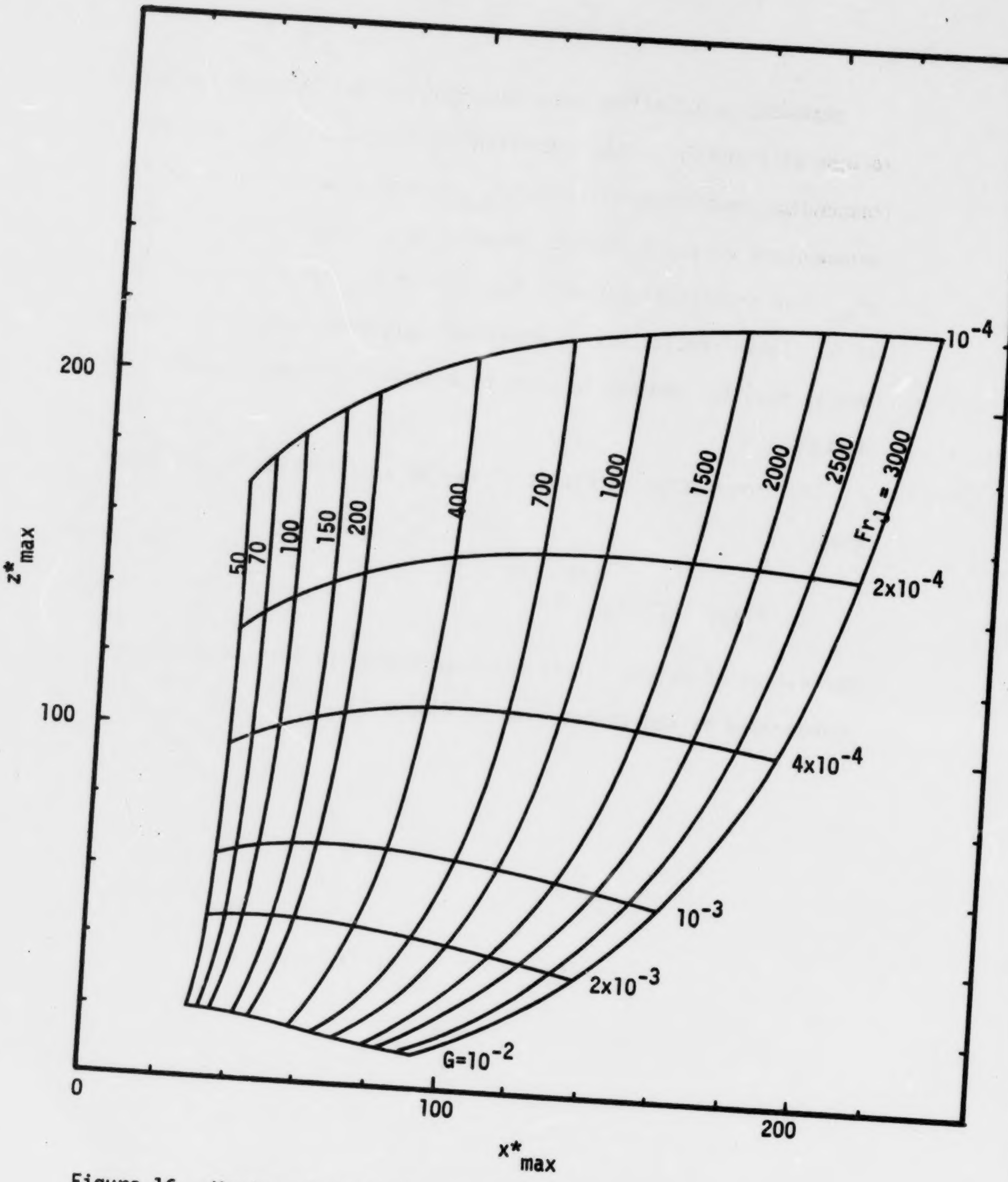


Figure 16. Maximum height of rise versus downstream distance for horizontal plume in linearly stratified environment.

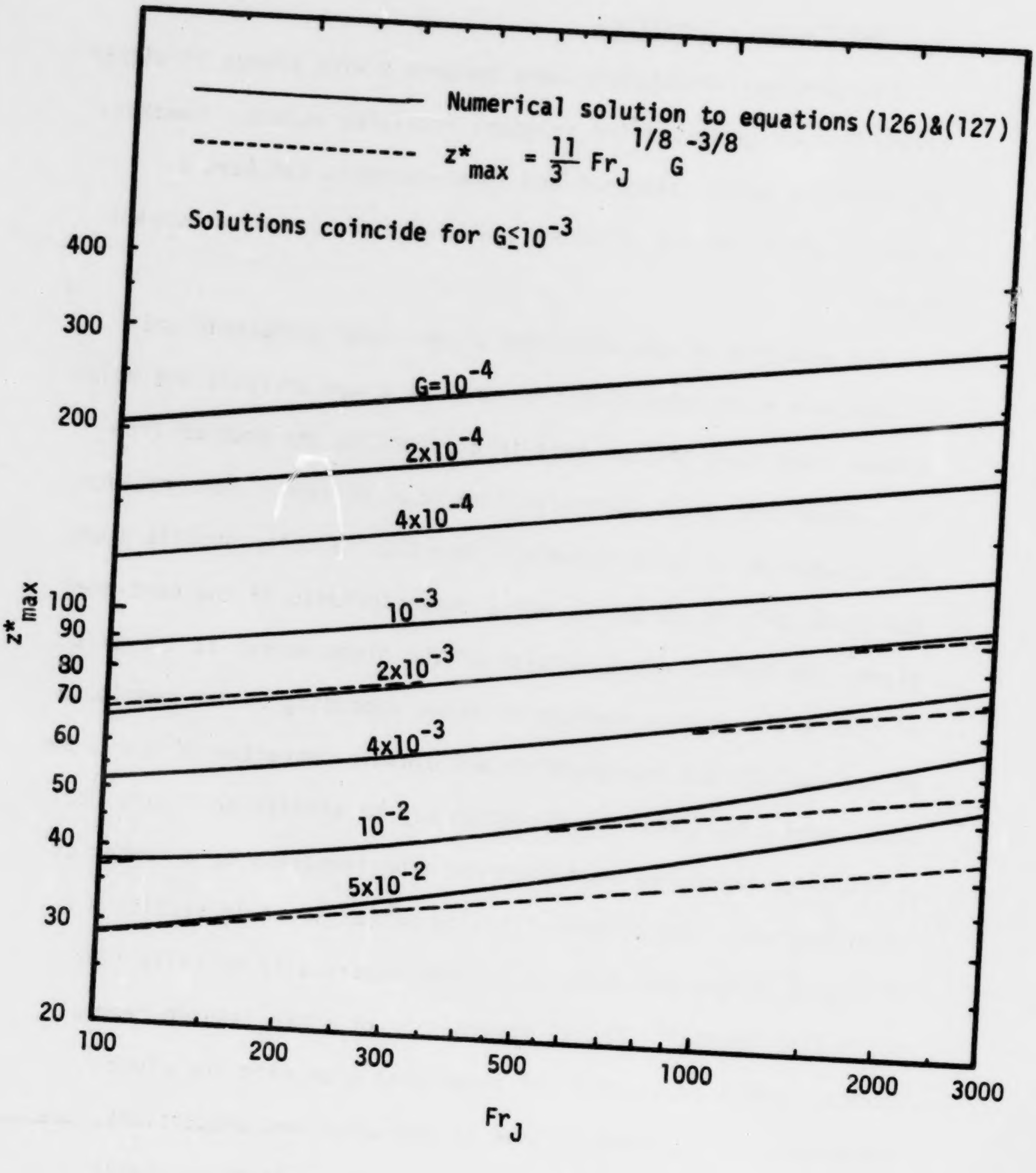


Figure 17. Maximum height of rise for a vertical plume in a linearly stratified environment.

Vertical Plumes - Crossflow

The previous discussions were concerned with plumes in either stratified or nonstratified stagnant receiving waters. However, the ocean is seldom stagnant and cross-currents can have a dramatic effect on the dilution and height of rise of a buoyant plume.

For analysis of the bent-over plume, simplifications and assumptions associated with the vertical plume analysis are maintained. One must realize that in approaching the problem from this viewpoint, gross approximations must be made. For instance, the assumption of an axisymmetric Gaussian velocity profile along the plume axis is certainly not a characteristic of the bent-over plume. In fact, a cross-section of the plume normal to the axis of flow will reveal a variety of forms depending on the magnitude of the current and the geometry and dynamic character of the plume. Under most conditions the situation may be similar to Figure 18, a situation which has been observed experimentally by a number of investigators. The counter-rotating motion shown in section A-A of Figure 18 has also been calculated numerically by Lilly (28) for a line thermal. Casual observation of smoke issuing from a chimney into a cross-wind has shown that when wind and plume conditions are adjusted in some as-yet-undefined proportions, the plume may disintegrate into two or even three distinguishable streams (Figure 19).

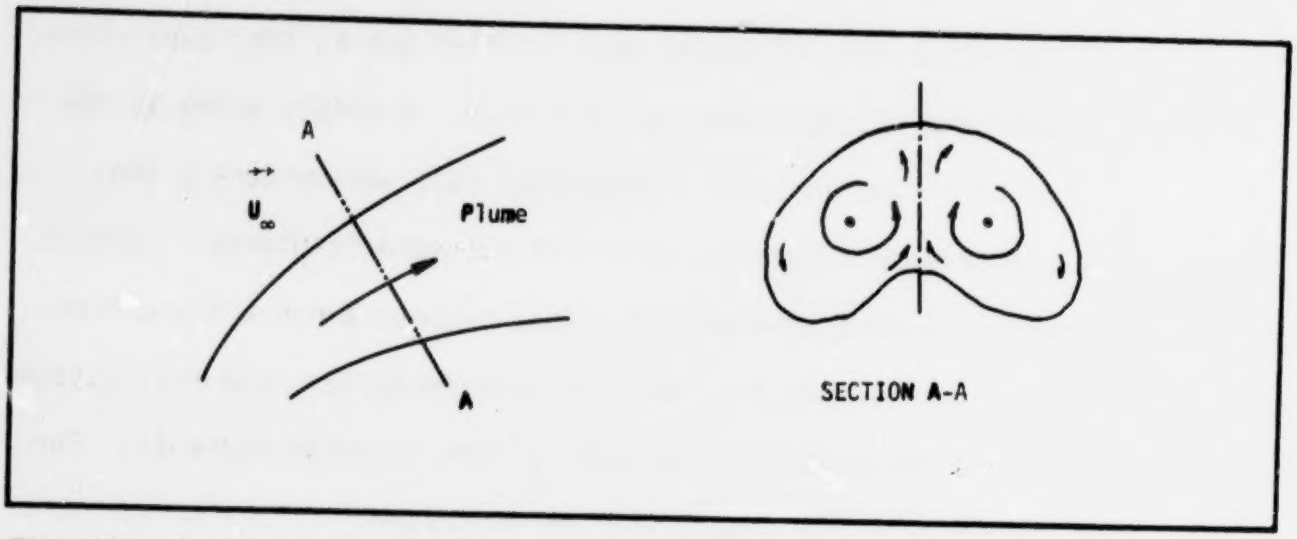

A
$\vec{U}_\infty$
Plume
A
SECTION A-A

Figure 18. Vortices associated with a bent-over plume.

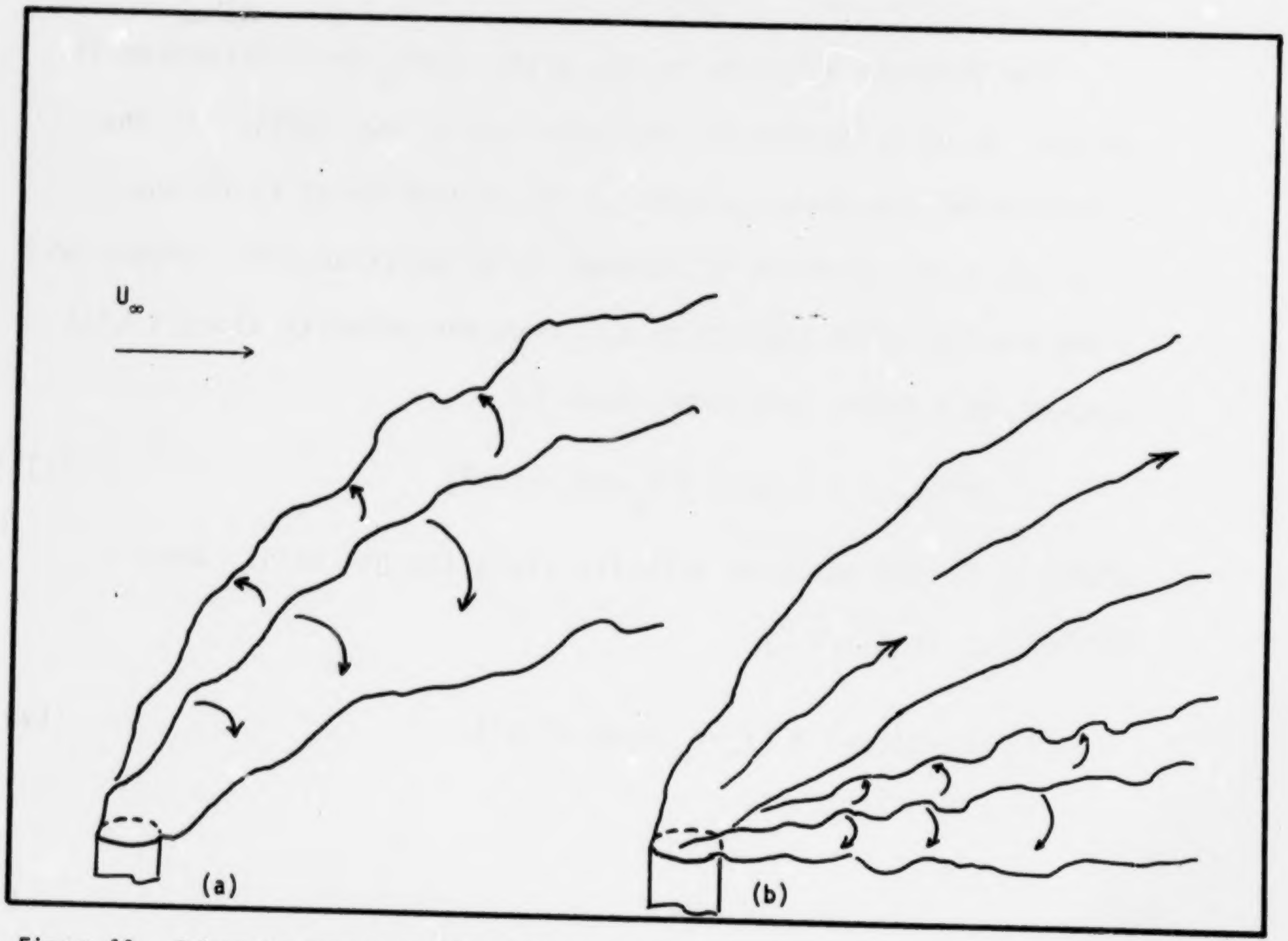


Figure 19. Schematic representation of smoke plume separating into (a) two or (b) three streams due to wind effects.

The amount of analytical work carried out on bent-over plumes is meager and in some cases quite flimsy - probably owing to the difficulty of the analysis. Priestley (42) was evidently the first to propose a working model for atmospheric plumes. Schmidt (46) has also done some work on smoke plumes, among a few others. However, Fan (14) presents what is considered here the most systematic approach to the problem of buoyant plumes in ocean currents. For this reason only Fan's work will be discussed.

Fan considered the problem of a vertical round port discharging buoyant pollutant to a homogeneous ocean with a uniform (top to bottom) cross-current (Figure 20).

The absolute velocity of the plume along the s-direction is assumed to be axisymmetric, and composed of two parts: 1) the component of the cross-current in the direction of s; 2) the relative velocity which is assumed to be Gaussian with respect to r and similar with respect to s. Thus the velocity along s with respect to a fixed reference frame is

$$U^*(s,r) = U_\infty \mathrm{Cos}\theta + U_m \exp[-r^2/b^2], \tag{131}$$

where U_m is the relative velocity along the jet axis. Density deficiency is given by

$$\rho_\infty - \rho(r,s) = [\rho_\infty - \rho_m]\exp[-r^2/b^2]. \tag{132}$$

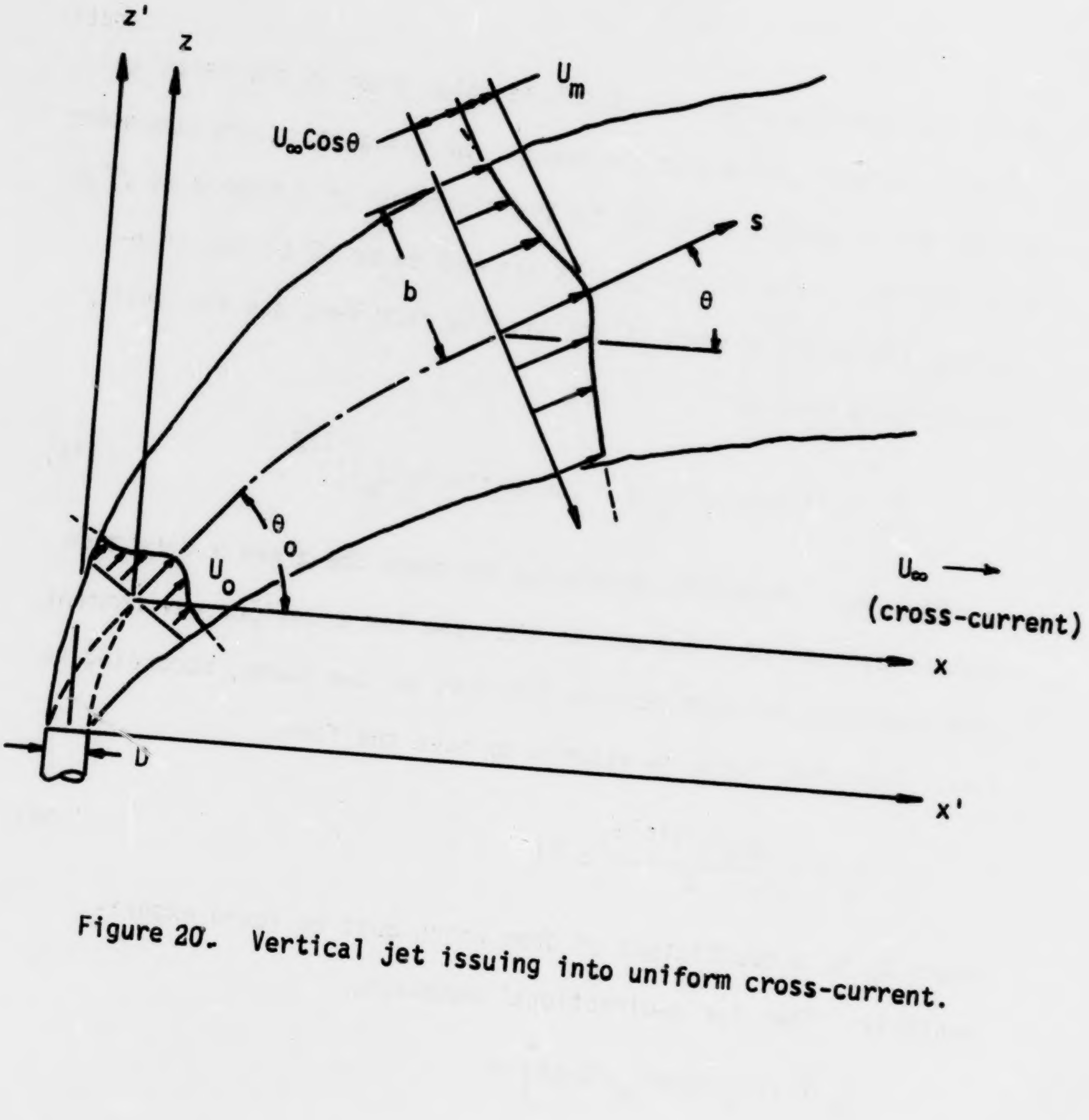

z'
z
U_m
$U_\infty \cos\theta$
s
b
θ
θ_o
U_o
U_∞ →
(cross-current)
x
D
x'

Figure 20. Vertical jet issuing into uniform cross-current.

The superscript * refers to the absolute velocity in the plume. Again, the governing equations must satisfy the conditions of continuity, momentum, and energy.

Continuity. Continuity is assumed to be satisfied by:

$$\frac{d}{ds}\int_A U^* dA = 2\pi\alpha b\left|\vec{U}_J-\vec{U}_\infty\right| ,$$

where the quantity under the absolute value sign is the relative velocity between jet and environment. By integrating the component of the environmental velocity in the direction of r from 0 to $\sqrt{2}$ b (the nominal radius of the jet is assumed to be $\sqrt{2}$ b) and integrating the Gaussian part of the profile from 0→∞, the continuity relationship becomes:

$$\frac{d}{ds}\left[b^2(2U_\infty \mathrm{Cos}\theta+U_m)\right] = 2\alpha b(U_\infty{}^2\mathrm{Sin}^2\theta+U_m{}^2)^{1/2} . \qquad (133)$$

Momentum. Momentum components for both the z and x direction must be considered. But unlike the case for a uniform environment, some compensation must be made for drag on the plume, according to Fan. This drag force is assumed to have the form:

$$F_d = C_d\left(\frac{\rho_\infty U_\infty{}^2\mathrm{Sin}^2\theta}{2}\right)2\sqrt{2}\,b , \qquad (134)$$

where C_d is a coefficient of drag which must be found experimentally. Then for x-directional momentum,

$$\frac{d}{ds}\left[\frac{b^2}{2}(2U_\infty \mathrm{Cos}\theta+U_m)^2\mathrm{Cos}\theta\right] =$$

$$2\alpha bU_\infty(U_\infty{}^2\mathrm{Sin}^2\theta+U_m{}^2)^{1/2} + \frac{C_d\sqrt{2}}{\pi}U_\infty{}^2 b\mathrm{Sin}^3\theta, \qquad (135)$$

and for z-directional momentum,

$$\frac{d}{ds}\left[\frac{b^2}{2}\,(2U_\infty \text{Cos}\theta+U_m)^2\text{Sin}\theta\right] = b^2 g\left(\frac{\rho_\infty-\rho_m}{\rho_\infty}\right)$$

$$-\frac{c_d\sqrt{2}}{2}\,U_\infty{}^2 b\ \text{Sin}^2\theta\text{Cos}\theta. \qquad (136)$$

Density deficiency. The relationship for density deficiency for a non-stratified medium is

$$\frac{d}{ds}\int_A U^*(\rho_\infty-\rho)dA = 0. \qquad (137)$$

Substituting the expressions (131) and (132) for U^* and ρ, and integrating from $0\rightarrow\infty$, yields

$$\frac{d}{ds}\,[b^2(2U_\infty\text{Cos}\theta+U_m)(\rho_\infty-\rho_m)] = 0. \qquad (138)$$

The geometric relationship between s-θ coordinates and x-z coordinates are given as

$$\frac{dz}{ds} = \text{Sin}\theta, \qquad (139)$$

and $$\frac{dx}{ds} = \text{Cos}\theta. \qquad (140)$$

Initial conditions are referred to the origin of the s-θ coordinate system and have the following values: $U_m(0)=U_J$, $b(0) =b_o\left(b=\frac{D}{\sqrt{2}}\right)$, $\rho_m(0)=\rho_J$, $\theta(0)=\theta_o$, $z=x=0$ at $s=0$. Equation (138) can be integrated immediately to yield:

$$b^2(2U_\infty\text{Cos}\theta+U_m)(\rho_\infty-\rho_m) = \text{constant}, \qquad (141)$$

but the five remaining differential equations ((133), (135), (136), (137), and (138)) must be solved simultaneously. Since the ambient medium is assumed non-stratified, concentration may be calculated using equation (141) with $(\rho_\infty-\rho_m)$ replaced by C_m. Fan carried out these computations numerically for various cases after normalization of equations and non-dimensionalizing parameters. One other point that should be mentioned here is that θ_o is no longer 90° from the horizontal where established flow begins.

To calculate θ_o at $s=s_o$, Fan uses the empirical relationship

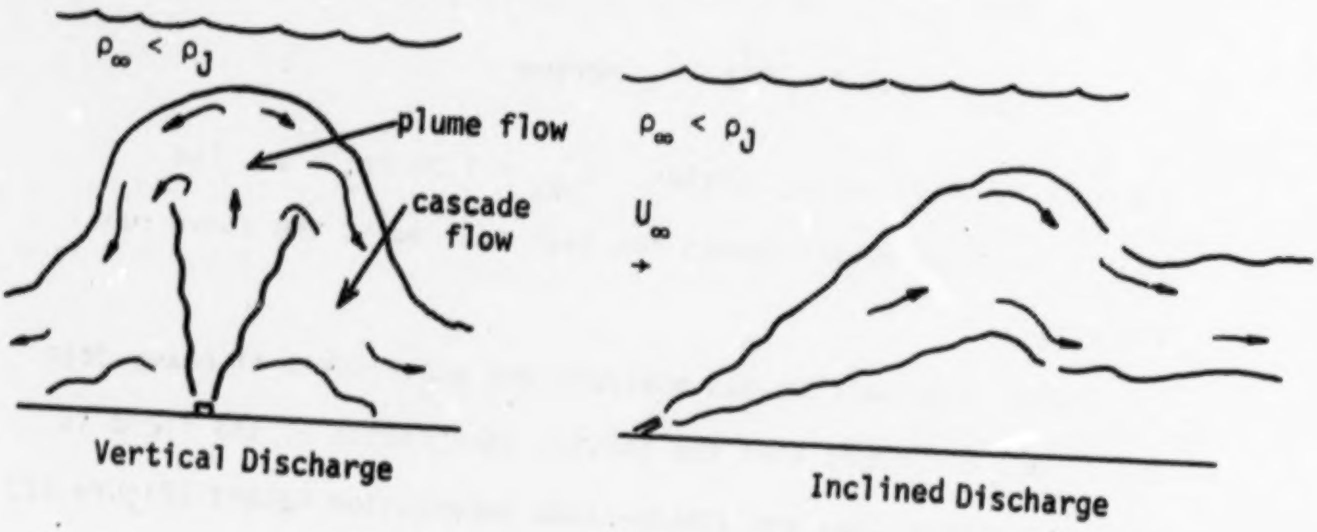
$$\theta_o = 90-110K' \qquad (142)$$

where K' is a function of s/D (see reference 14, page 94).

A very brief and incomplete summary of Fan's work has been presented here. Reference 14 should be consulted for the detailed theory and results.

Negatively Buoyant Plumes

Plumes that have negative buoyancy result when heavier fluid is discharged into lighter receiving water such as water with heavy salt concentration rejected from a desalinization plant. The dynamics are essentially the same as in the case of a buoyant plume, except the sign of the buoyancy term is reversed. However, in the case of the vertical plume the heavier fluid will cascade, causing interaction between the initially formed plume and the cascade flow. For cases where the fluid issues from an inclined port, and in the same general direction as the ambient current, this interaction will not take place. Figure 21 illustrates typical negative buoyancy flows.



Vertical Discharge

Inclined Discharge

Figure 21. Typical flow patterns for plumes with negative buoyancy.

Analysis similar to the positively buoyant plume should hold for the inclined discharge if the sign of the buoyancy term is changed in the governing differential equation. However, with the vertical discharge there is some question whether an adequate similarity solution can be expected because of the interaction between flows. It has been the practice thus far to assume that there is no mixing between the two flows. In this way solutions for the positive buoyancy plume can be fashioned into solutions for the heavy plume.

Cederwall (11) gives the following comparisons of theoretical ceiling heights for heavy fluids discharged vertically into homogeneous environments:

$$z_{max} = 1.94\ Fr_J^{1/2}D \qquad \text{Abraham}$$

$$z_{max} = 1.85\ Fr_J^{1/2}D \qquad \text{Priestly \& Ball} \qquad (143)$$

$$z_{max} = 1.60\ Fr_J^{1/2}D \qquad \text{Morton}$$

The work by Trent et al. yields: $z_{max} = 1.80\ Fr_J^{1/2}D$. The length for flow establishment has been ignored in the above relationships.

Turner (54) carried out analysis and experiments of heavy jets and plumes and found that the initial penetration of the fluid is somewhat higher than the steady-state penetration height (Figure 22).

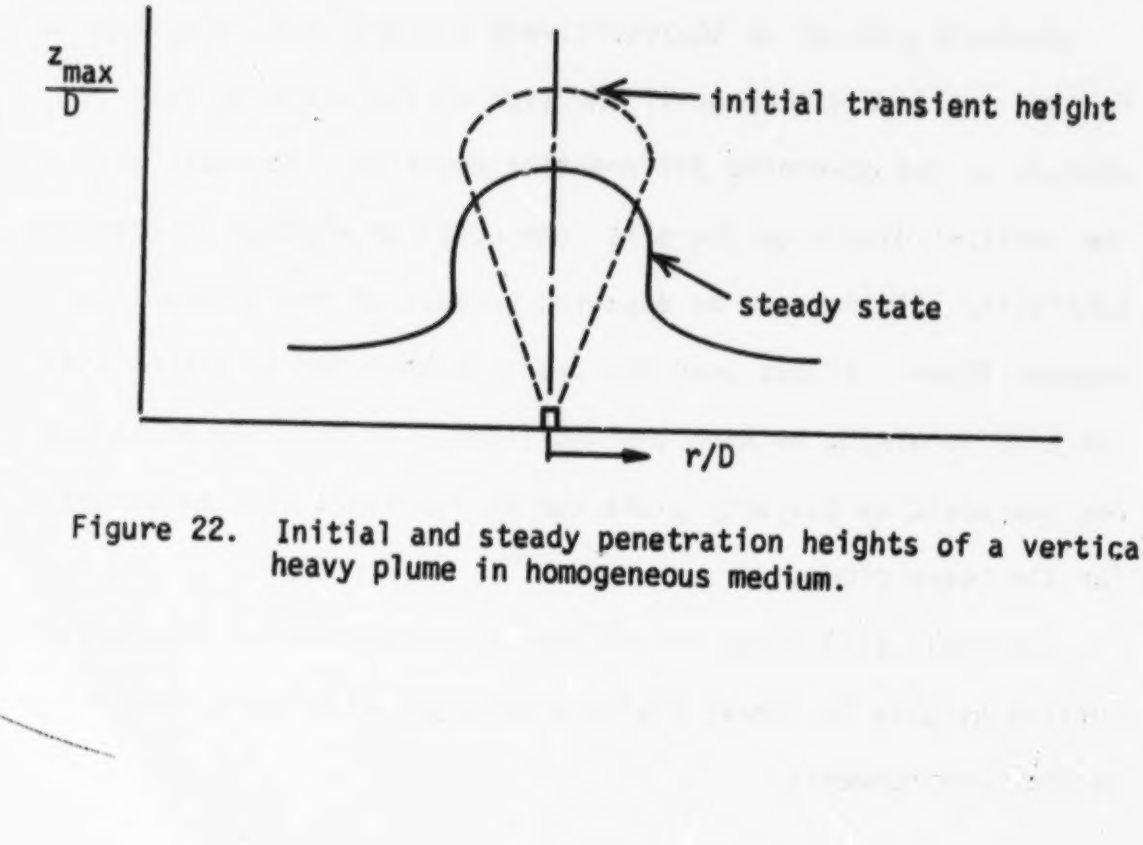


Figure 22. Initial and steady penetration heights of a vertical heavy plume in homogeneous medium.

Turner's experimental results (for comparison with (143)) yield

$$z_{max} = 1.74\ Fr_J^{1/2} D \text{ (steady state)}$$

and $$z_{max} = 2.48\ Fr_J^{1/2} D \text{ (initial transient).} \qquad (144)$$

The initial transient penetrates higher apparently because there is no cascading flow to interact with, as is the case of steady state. Turner's theoretical penetration height for steady state is:

$$z_{max} = 2.9 Fr_J^{1/3} D. \qquad (145)$$

Horizontal Surface Discharge

There may be benefit in discharging certain pollutants at the sea surface. For instance, thermal interchange with the atmosphere would hasten the dissipation of heat and lower the overall heat-up of the local sea. Also, shoreline discharge is economically attractive because the cost of constructing an expensive marine discharge line is eliminated.

Very little analysis has been carried out for this type of discharge. The dynamics of the pollution field are again characterized by a zone of initial dilution and a zone of drift flow. However, the zone of initial dilution is influenced primarily by the discharge momentum. If the pollutant is lighter than the receiving water, a gravitational spread must be considered along with the turbulent diffusion.

In some cases, such as low velocity canal discharge of power plant reject heat, initial momentum of the coolant may play only a minor role on the overall pollution field dynamic behavior. In such instances, the pollutant field dynamics must be treated by methods discussed in chapters V and VI.

Jen et al. (24) have carried out analytical and experimental investigations for horizontal surface jets having initial densimetric Froude numbers in the range of 324 to 32,400. The results of this study showed that the surface temperature distribution could be approximated by

$$\frac{T-T_\infty}{T_m-T_\infty} = \exp\left[-3Fr_J^{1/4}\left(\frac{y}{x}\right)^2\right], \qquad (146)$$

for $x/D<100$. In the above equation, T_∞ is the receiving water temperature, T_m is the jet centerline temperature, x is the coordinate along the jet axis, and y is normal to the jet axis.

For values of $x/D<100$ the centerline temperature is expressed by

$$\frac{T_m-T_\infty}{T_J-T_\infty} = 7.0\left(\frac{D}{x}\right), \qquad (147)$$

where T_J is the effluent temperature.

The above equations were developed in a manner similar to the methods used by Abraham. Also, these equations are only valid where initial momentum is dominating the flow dynamics (large Fr_J).

It is doubtful that the analysis presented by Jen et al. could be applied to practical cases with a great deal of confidence, because the range of Froude numbers covered by the investigation does not correspond to most practical situations. For instance, the densimetric Froude number for canal discharge of hot water may be on the order of 10 or less.

Zeller (61) considered the surface jet from a viewpoint similar to Fan's (14) analysis of the buoyant plume. However, unlike the work of Jen et al., Zeller incorporates the effects of cross-currents and wind stress.

Hayashi and Shuto (20) also studied the problem of hot water spreading over colder receiving water. These authors considered the case of a very low initial densimetric Froude number ($Fr_J \approx 1$) and ignored the non-linear terms in the equations of motion. Hence, this work is not defined in the category of jet flow, but is discussed in Chapter V under "Surface Drift of a Buoyant Pollutant."

ZONE OF DRIFT FLOW

The zone of drift flow is characterized by the pollutant field drifting with the oceanic currents nearly as if it were a part of the oceanic velocity field. In the drift flow regime the pollutant field continues to dilute and disperse into the oceanic environment primarily by two possible mechanisms. The first of these mechanisms is turbulent transport or eddy mixing; the second is a density disparity between the pollutant flow field and the environment. The first mechanism, turbulent mixing, will always be present in the ocean, but spreading by virtue of a density disparity may not be. Lateral spread of a pollution field under the action of gravity is nearly entirely a surface phenomenon* and is characterized by the polluted stream from an outfall reaching the ocean surface without sufficient dilution so that it remains on the surface, with positive buoyancy (or on the bottom with negative buoyancy). As a result of the density disparity, the action of gravity causes the lighter pollution to spread laterally over the heavier receiving water. At the same time lateral eddy diffusion is also acting. Depending on the intensity of vertical turbulence, vertical dispersion may not be significant. Figure 23 illustrates the different types of drift flow schematically for a submerged horizontal discharge of buoyant fluid in a density-stratified receiving water.

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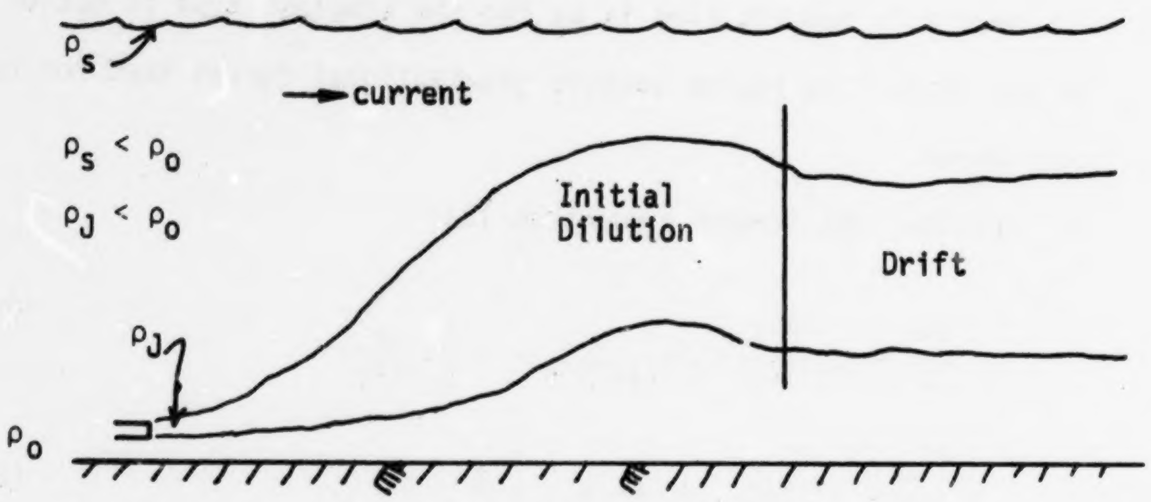
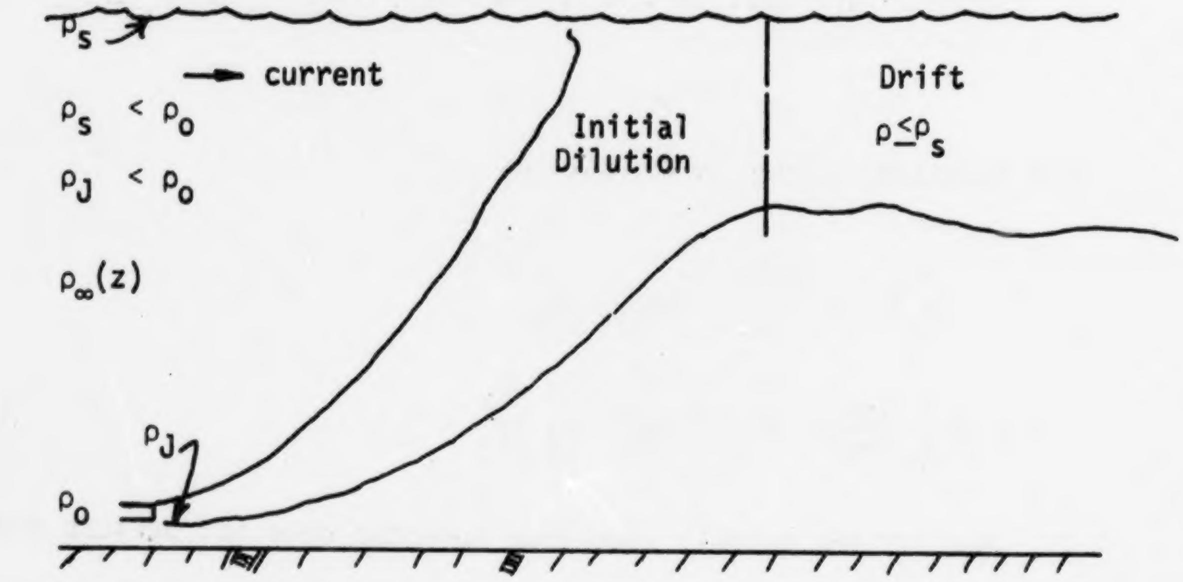
* Cederwall (11) points out that a submerged lateral density spread will occur under certain conditions and may, in fact, become an important consideration in some locales.


ρ_s
current
$\rho_s < \rho_o$
$\rho_J < \rho_o$
Initial Dilution
Drift
ρ_J
ρ_o

A) Submerged Drift Flow


ρ_s
current
$\rho_s < \rho_o$
$\rho_J < \rho_o$
$\rho_\infty(z)$
Initial Dilution
Drift
$\rho \leq \rho_s$
ρ_J
ρ_o

B) Surface Drift Flow

Figure 23. Two types of drift flow resulting from submerged horizontal discharge of buoyant fluid in a density stratified ocean with current.

Submerged or Neutrally Buoyant Drift Flow

Neutrally buoyant flow is by far the simplest type of motion in the drift flow regime because gravitational forces need not be considered.

Consider the momentum equation (8):

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + 2e_{ijk}\Omega_j U_k = - \frac{1}{\rho_o} \frac{\partial p}{\partial x_i}$$

$$+ \left[\frac{\rho_\infty - \rho}{\rho_o} - 1\right] g\delta_{i3} + \frac{\partial}{\partial x_j} \varepsilon_{\underline{i}\underline{j}} \frac{\partial U_i}{\partial x_j} . \qquad (8)$$

Since we are dealing with a neutrally buoyant pollutant field,

$$\rho = \rho_\infty$$

and equation (8) may be written as

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + 2e_{ijk}\Omega_j U_k$$

$$= -\frac{1}{\rho_o} \frac{\partial p}{\partial x_i} - g\delta_{i3} + \frac{\partial}{\partial x_j} \varepsilon_{\underline{i}\underline{j}} \frac{\partial U_i}{\partial x_j} . \qquad (148)$$

Now, if the outfall discharge momentum does not perturb the oceanic motion to any appreciable extent, solution of the above equation is independent of the pollutant field.

The pollutant dispersion is governed by the mass diffusion equation (10),

$$\frac{\partial C}{\partial t} + U_j \frac{\partial C}{\partial x_j} = \frac{\partial}{\partial x_j} \left[\mathcal{D} + \varepsilon_{M\underline{j}}\right] \frac{\partial C}{\partial x_j} + \dot{m} , \qquad (149)$$

where C is concentration of the pollutant.

The quantity $\dot{m}$ represents a source or sink that is used to account for BOD decay, growth of bacteria, sedimentation, etc. For oceanic flow $\mathcal{D} << \varepsilon_{Mj}$, so that

$$\frac{\partial C}{\partial t} + U_j \frac{\partial C}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\varepsilon_{M\underline{j}} \frac{\partial C}{\partial x_j}\right] + \dot{m}, \tag{150}$$

where ε_{Mj} is a function of the flow field and is not a fluid property. To solve equation (150) for C, the quantities U_m and ε_{Mj} must be known.

The velocity components, U_j, may be calculated using equation (148), or they may be determined by oceanographic measurements. The eddy diffusion coefficients, ε_{Mj}, are determined by measurement and may be constant or assumed to have some functional form (to be discussed later).

At any rate, the problem of calculating C from equation (150) is considerably simplified since the velocity field is assumed to be known beforehand. This situation contrasts the case of a buoyant surface drift where the momentum equation (8) must be considered simultaneously with equation (150) because of the interaction between pollutant field concentration and momentum (i.e. density disparity, $\rho_\infty - \rho$, causing a gravity spread of the field).

Now if the coordinate system is assumed to move along with the ocean current, then the convective terms in equation (150) vanish and

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x_j} \varepsilon_{M\underline{j}} \frac{\partial C}{\partial x_j} + \dot{m}. \tag{151}$$

If the ocean has a steady uniform velocity field, U_j, and the pollutant has been issuing from a continuous source long enough for steady-state conditions to prevail at downstream points, then

$$\frac{\partial C}{\partial t} = 0. \tag{152}$$

Hence, equation (150) becomes

$$U_j \frac{\partial C}{\partial x_j} = \frac{\partial}{\partial x_j} \left[\varepsilon_{M\underline{j}} \frac{\partial C}{\partial x_j} \right] + \dot{m} \; . \tag{153}$$

For the case of homogeneous, isotropic turbulence ε_{Mj} is constant with respect to spatial coordinates so that from equation (151),

$$\frac{\partial C}{\partial t} = \varepsilon_M \frac{\partial^2 C}{\partial x_j \partial x_j} + \dot{m}, \tag{154}$$

or, from equation (153),

$$U_j \frac{\partial C}{\partial x_j} = \varepsilon_M \frac{\partial^2 C}{\partial x_j \partial x_j} + \dot{m} \; . \tag{155}$$

Finally, if the substance is conserved, $\dot{m}=0$, then

$$\frac{\partial C}{\partial t} = \varepsilon_M \frac{\partial^2 C}{\partial x_j \partial x_j} \tag{156}$$

and

$$U_j \frac{\partial C}{\partial x_j} = \varepsilon_M \frac{\partial^2 C}{\partial x_j \partial x_j} \; . \tag{157}$$

Comments on the Eddy Transport Coefficient, ε_{Mj}

In order to solve the above diffusion equations analytically, the eddy diffusion coefficient, ε_{Mj}, must have some functional form. Obviously, the simplest case is that of a homogeneous isotropic turbulence field where $\varepsilon_{Mj}=\varepsilon_M$, a constant. However, it may not be reasonable to assume the ocean turbulence is either homogeneous or isotropic. For example, wind stress at the sea surface causes ocean turbulence which diffuses with decreasing intensity to lower depths. Also, density stratification depresses the component of vertical mixing so that lateral turbulent transport is much greater than the vertical transport.

Another complicating factor is that as the plume grows in size, larger and larger scales of turbulence take part in the diffusion process. In the ocean, turbulence of all scales can exist, from the isotropic microscales to the highly anisotropic macroscales which are limited in size by solid boundaries and the sea surface. Various authors have attempted to account for this phenomena by assuming that the diffusion coefficient is a function of the plume size and/or is a function of diffusion time.

In the open sea (see Reference 36) measurements have shown that the lateral eddy diffusion coefficient is proportional to $L^{4/3}$, where L is the scale size (the width of a dye plume, for instance). However, in the open sea very large scales of turbulence are possible, whereas in a coastal region the scales would be limited in size by the coastline and bottom effects.

Orlob (38) carried out various eddy diffusion studies in open channel flow. From experiment he found that lateral dispersion patterns would not grow indefinitely according to the 4/3 power law, but would become parabolic in shape as governed by the largest sized eddies participating in the diffusion process. These findings indicate that when the pollution field is smaller than the largest eddies, the eddy diffusion coefficient is proportional to $L^{4/3}$, but when the plume grows to a size larger than the largest eddies, the eddy diffusion coefficient is constant (actually there is a transition zone separating these two effects).

Cederwall (11) argues that in shallow coastal waters the growth of turbulent eddies is limited, and at a distance not too far from the source, the plume will grow to a size where the maximum possible sized eddies are affecting the diffusion. Thereafter the eddy diffusion coefficient is essentially constant - at least with regard to maximum scale sizes. For this reason Cederwall maintains that it is more reasonable to base diffusion models on a constant diffusion coefficient. As an example, Harremoes (18) has found from radioactive tracer studies off Hälsingborg, Sweden, that the coefficient of lateral eddy diffusion was essentially constant (ε_M lateral $\simeq 0.5$ meters²/sec) in these waters. He also found that the vertical eddy diffusion could be approximated by

$$\varepsilon_{Mz} \simeq 5\text{x}10^{-5} \, \frac{\partial \rho}{\partial z}^{-2/3} \quad \text{meters}^2/\text{sec},$$

for the same region. Foxworthy et al. (15) also questions the validity of using the 4/3 power law in coastal waters, based on measurements off Southern California which indicated considerable disagreement with calculations based on the 4/3 power law.

Ozmidov (39) has proposed a function, f(L/H), which corrects the 4/3 power law for shallow coastal water. Using this correction factor Ozmidov writes the coefficient of eddy diffusion as

$$\epsilon_M(L,L/H) = kL^{4/3}f(L/H),$$

where k is a proportionality constant. However, this proposal has apparently not received general acceptance. A good summary of both lateral and vertical eddy transport coefficient models is given by Koh and Fan (25).

The eddy diffusion coefficients are fundamental to the solution of the mass diffusion equation. At present there is no general agreement as to the functional form of these coefficients for applications in coastal water. Obviously, a good deal of research is needed concerning large-scale turbulence in shallow coastal waters.

Point Source Models

A number of solutions to the diffusion equation are possible based on boundary effects and the assumed form of the diffusion coefficient. A few of these proposed diffusion models are presented here for a conservative pollutant ($\dot{m} = 0$).

Fickian diffusion models* are based on a constant value of eddy diffusion which are not necessarily isotropic.

For the case of an instantaneous release of an amount, M, of pollutant in a uniform current having velocity components U_∞, V_∞, and W_∞ in the x,y, and z directions, respectively, the concentration at a point is given by

$$C(x,y,z,t) = \frac{M}{(4\pi)^{3/2}(\varepsilon_{Mx}\varepsilon_{My}\varepsilon_{Mz})^{1/2}t^{3/2}} \exp - \left[\frac{(x-U_\infty t)^2}{4\varepsilon_{Mx}t} + \frac{(y-V_\infty t)^2}{4\varepsilon_{My}t} + \frac{(z-W_\infty t)^2}{4\varepsilon_{Mz}t}\right]. \tag{158}$$

For a fixed source, a continuous volumetric discharge, q, of pollutant into a uniform current flowing in the x-direction only, results in

$$C(x,y,z) = \frac{q}{4\pi(\varepsilon_{My}\varepsilon_{Mz})^{1/2}x} \exp - \left[\frac{y^2U_\infty}{4\varepsilon_{My}x} + \frac{z^2U_\infty}{4\varepsilon_{Mz}x}\right]$$

In the above equation, turbulent transport in the x-direction is ignored; specifically, $\varepsilon_x/U_\infty x \ll 1$ is assumed.

To account for the sea surface boundary

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* This work has been adapted from Reference 11.

$$C(x,y,z) = \frac{q}{2\pi(\varepsilon_{My}\varepsilon_{Mz})^{1/2}x} \exp - \left[\frac{y^2U_\infty}{4\varepsilon_{My}x} + \frac{z^2U_\infty}{4\varepsilon_{Mz}x}\right] \qquad (160)$$

where z=0 may be take as the sea surface (the plane of reflection).

Models Based on a Variable Diffusion Coefficient

For a conservative pollutant, the simple radial flow diffusion equation takes the form

$$\frac{\partial C}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[\varepsilon_{Mr}\, r\, \frac{\partial C}{\partial r}\right]. \qquad (161)$$

Specifically, this equation is used to describe the diffusion of the instantaneous release of an amount M of pollutant. The pollutant is assumed to spread only in a radial plane and any solid boundary effects are ignored. Different models may be derived from equation (161) by assuming different functional forms for ε_{Mr}. Okubo (37) has generalized these various solutions by assuming that the eddy transport coefficient ε_{Mr} may be a function of space and time, defined functionally by

$$\varepsilon_{Mr} = \varepsilon_{Mo} f(t) r^m. \qquad (162)$$

Then equation (161) may be written as

$$\frac{\partial C}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[\varepsilon_{Mo} f(t) r^{m+1} \frac{\partial C}{\partial r}\right] \qquad (163)$$

where ε_{Mo} and m are constants and f(t) is a function of time related to the concentration distribution.

The general solution to (163) is:

$$C(r,t) = \frac{(2-m)M \quad \exp - \left[\frac{r^{(2-m)}}{(2-m)^2 \varepsilon_{Mo} \psi(t)}\right]}{2\pi(2-m)^{4/(2-m)} \Gamma(\frac{2}{2-m}) \varepsilon_{Mo}{}^{2/(2-m)} [\psi(t)]^{2/(2-m)}} \qquad (164)$$

for $0 \leq m \leq 2$. The symbol Γ denotes the Gamma function and

- $\psi(t) = \int_0^t f(\xi) d\xi$
- $C(r,o) = m\delta(r)/\pi r$ $(\delta(r)$ = Dirac delta function$)$
- $\lim_{r \to \infty} C(r,t) = 0, t \geq 0$
- $C(r,t) = 0$, for $r<0$, $t \geq 0$
- $\int_0^\infty C(r,t) 2\pi r dr = M$, for $t \geq 0$

Table I summarizes the models derived from equation (163) (or (164)) proposed by various authors.

Brooks (reference 60) has presented solutions based on the assumption that the coefficient of eddy diffusion is a function of the pollution field width, L, alone. The results of this work are:

$$C_m (x) = \frac{M}{2\sqrt{6\pi\alpha_1}} \left(\frac{x}{U_\infty}\right)^{-1}, \ \varepsilon = \alpha_1 L,$$

$$C_m (x) = \sqrt{\frac{3}{256\pi}} \frac{M}{\alpha_2^{3/2}} \left(\frac{x}{U_\infty}\right)^{-3/2}, \ \varepsilon = \alpha_2 L^{4/3},$$

TABLE 1

RADIAL DIFFUSION MODELS FROM OKUBO (37)

	f(t)	m	Experimental Parameter	Proposed by:
$C(r,t)=\dfrac{M\exp[-\frac{r^2}{4\epsilon_{Mo}t}]}{4\pi\epsilon_{Mo}t}$	1	0	ϵ_{Mo}	"Fickian"
$C(r,t)=\dfrac{M\exp[-\frac{r}{pt}]}{2\pi p^2t^2}$	1	1	p:Diffusion Velocity	Joseph & Sender
$C(r,t)=\dfrac{M\exp[\frac{-r^{2/3}}{\alpha t}]}{6\pi\alpha^3t^3}$	1	4/3	α:Energy Dissipation	Ozmidov
$C(r,t)=\dfrac{M\exp[\frac{-r^2}{\omega^2t^2}]}{\pi\omega^2t^2}$	t	0	ω:Diffusion Velocity	Pritchard & Okubo
$C(r,t)=\dfrac{M\exp[\frac{-r^{4/3}}{\alpha^2t^2}]}{(3/4)\pi^{3/2}\alpha^3t^3}$	t	2/3	α:Energy Dissipation	Okubo
$C(r,t)=\dfrac{M\exp[\frac{-r^2}{\beta^3t^3}]}{\pi\beta^3t^3}$	t^2	0	β:Energy Dissipation	Obukhov

where α_1 and α_2 are determined by experiment and L is the width of the pollution field.

Line Source Models

Brooks (10) considered the dispersion of matter from an ocean outfall line with pollutants issuing from a number of ports along the line. For analytical purposes, he considered the outfall system a steady line source issuing horizontally in a uniform stream U_∞. By neglecting vertical and longitudinal eddy transport, Brooks writes equation (153) as

$$U_\infty \frac{\partial C}{\partial x} = \frac{\partial}{\partial y}\left(\varepsilon_M \frac{\partial C}{\partial y}\right) + \dot{m}. \qquad (165)$$

Figure 24 illustrates the plan view of the idealized physical system.

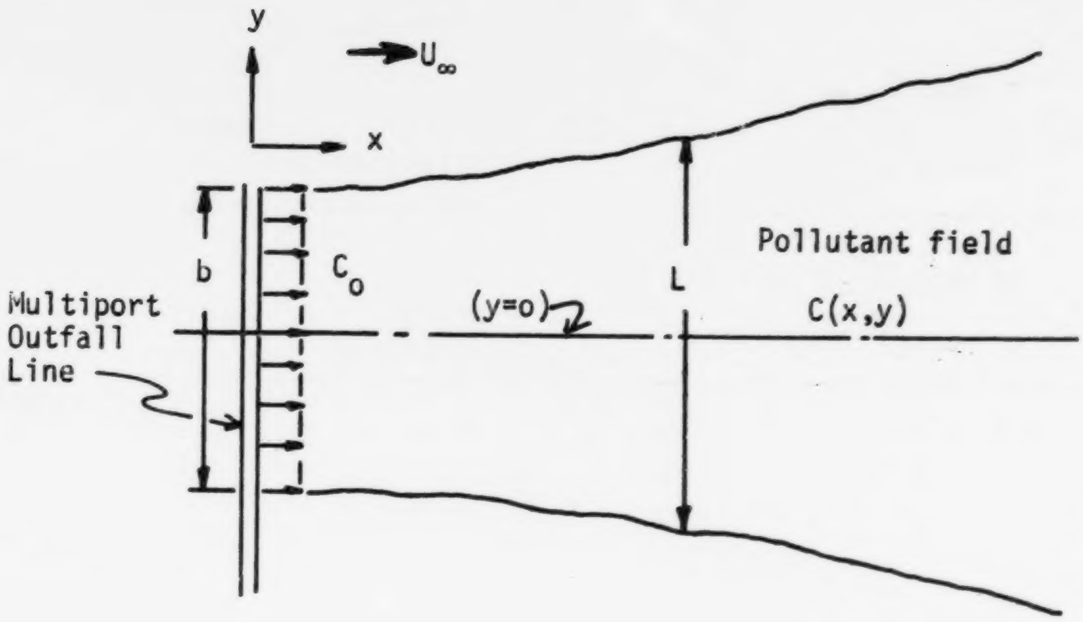

Figure 24. Brooks' line source model (for ported outfall line).

Brooks assumes that for biological and/or solid pollutants the die-off rate and/or sedimentation are proportional to the concentration, so that

$$\dot{m} = kC.$$

This assumption has a net effect on the final solution of multiplication of C by $\exp(-kx/U_\infty)$. Since this is the only effect, $\dot{m}$ can be ignored in solving the diffusion equation, keeping in mind that the solution must be multiplied by this exponential factor. Brooks also maintains that equation (165) may be solved considering ε_M as a function of x alone. Thus, equation (165) may be written as

$$\frac{\partial C'}{\partial x} = \frac{\varepsilon_M(x)}{U_\infty} \frac{\partial^2 C'}{\partial y^2} , \qquad (166)$$

where C' is defined by $C=C' \exp(-kx/U_\infty)$.

Now let $\varepsilon_M(x) = \varepsilon_{Mo} f(x)$ where $f(o) = 1$, and define x' such that

$$dx' = f(x)dx.$$

The quantity ε_{Mo} is the eddy diffusion coefficient at $x = o$. Then equation (166) becomes:

$$\frac{\partial C'}{\partial x'} = \frac{\varepsilon_{Mo}}{U_\infty} \frac{\partial^2 C'}{\partial y^2} . \qquad (167)$$

The solution for (167) is

$$C'(x,y) = \frac{C_o}{2\sqrt{\pi\varepsilon_{Mo} t'}} \int_{-b/2}^{b/2} \exp\left[\frac{-(y-y')^2}{4\varepsilon_{Mo} t'}\right] dy' \qquad (168)$$

where C_o is the concentration of the plume as it enters the zone of drift flow, and $t' = x'/U_\infty$.

Brooks defines the field width as

$$L = 2\sqrt{3}\,\sigma$$

where σ, the standard deviation, is defined by

$$\sigma^2 = \frac{1}{C_o b}\int_{-\infty}^{\infty} y^2 C(x,y)\,dy. \tag{169}$$

Brooks evaluates the solution for three different assumptions of ε_M. These cases are listed in Table II. In Table II, β is defined by:

$$\beta = 12\varepsilon_{Mo}/U_\infty b.$$

Yudelson (60) discusses the previous diffusion equation solutions in somewhat more detail than given here, and he also compares theory with results from several experimental investigations (primarily dye-patch experiments). These comparisons will not be discussed in detail here except to point out that according to Yudelson, Brooks' solution using the 4/3 power law seems to yield reasonably accurate results. He also notes that Brooks' solutions have been used in designing a number of ocean outfalls in Southern California which are performing satisfactorily.

TABLE 2

LINE SOURCE SOLUTIONS

$\varepsilon(x)$	$\frac{L}{b}$	x'	$C_{max}(x)$
ε_{Mo}	$(1 + 2\,\beta\,\frac{x}{b})^{1/2}$	x	$C_o \text{ erf}\left\{\sqrt{\frac{3}{4\beta x/b}}\right\}\exp(-kt)$
$\varepsilon_{Mo}(\frac{L}{b})$	$(1 + \beta\,\frac{x}{b})$	$\frac{b}{2\beta}\,[(1 + \beta\,\frac{x}{b})^2 - 1]$	$C_o \text{ erf}\left\{\sqrt{\frac{3/2}{(1+\beta\,\frac{x}{b})^2 - 1}}\right\}\exp(-kt)$
$\varepsilon_{Mo}(\frac{L}{b})^{4/3}$	$(1 + \frac{2}{3}\,\beta\,\frac{x}{b})^{3/2}$	$\frac{b}{2}\,[(1 + \frac{2}{3}\,\beta\,\frac{x}{b})^3 - 1]$	$C_o \text{ erf}\left\{\sqrt{\frac{3/2}{(1+2/3\;\beta\,\frac{x}{b})^3 - 1}}\right\}\exp(-kt)$

$t = x/U_\infty$

Surface Drift of a Buoyant Pollutant

A buoyant pollutant not only disperses by the action of turbulent mixing, as in the case of neutrally buoyant drift, but also spreads because of the density disparity between the plume and surrounding water. This means that the velocity terms in equation (9) (or (150)) are not independent of the plume temperature (or concentration) and cannot be established beforehand through oceanographic measurement or by independent solution of the equations of motion. This complication leads to the necessity of solving the equations of motion and heat diffusion (or mass diffusion) simultaneously.

Consequently, the solution to the buoyant drift problem is considerably more complicated than the neutrally buoyant drift case. In fact, it is worthwhile investigating ways to determine if for a specific case the gravity spread is negligible compared to the turbulent diffusion. If this can be established, then methods used for neutrally buoyant drift can be employed.

However, for the general case, gravitational effects which cause the plume to spread laterally cannot be ignored and we are left with a difficult problem of solving equations (8) and (9) (or (150)) simultaneously.

Density profiles may show essentially a discontinuity between the pollution field and the receiving water, giving rise to a two-layer flow, or the profiles may be smooth, with the density becoming a minimum at the surface. Whether or not the layering effect prevails depends on the density and miscibility of the pollutant and the vertical eddy mixing of the receiving water. In either case gravity will tend to spread the lighter material laterally over the heavier receiving water, and air-sea interactions may also have a significant effect on the dispersion of the pollutant.

Apparently, there has been very little analytical work done for this type of flow, probably owing to the complexity of the mathematics. Cederwall (11) presents a calculational method from Larsen and Sorensen (26, not seen) for predicting the spread of layered flow under the action of gravity only, in a current with velocity U_∞ (see Figure 25). The procedure is based on the main assumption that the two layers remain perfectly identified with no frictional effects between layers, which implies: 1) no vertical mixing, and 2) a homogeneous pollution field. Also, only lateral spreading is considered (normal to current flow). Thus, the velocity of advance of the pollution front is given entirely by conditions at the front. By continuity of the pollutant

$$q_o = 2U_\infty b(x)\bar{h}(x), \tag{170}$$

where q_o designates volumetric flow of the pollutant from the source, U_∞ is the current velocity, $b(x)$ is the half-width of the field a distance x downstream from where the pollutant appears at the surface, and $\bar{h}(x)$ is the average cross-sectional depth of the

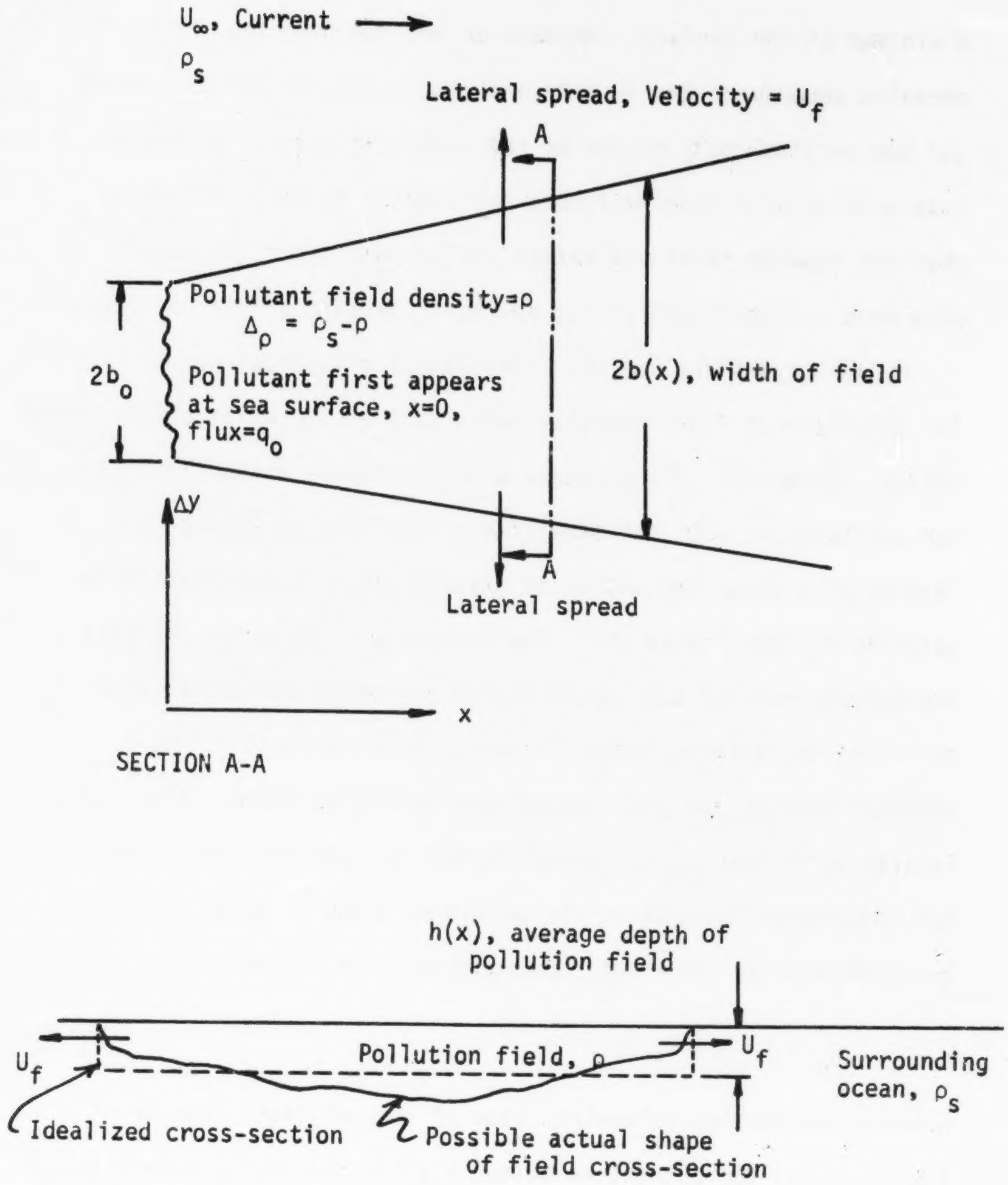
U∞, Current
ρs
Lateral spread, Velocity = Uf
A
Pollutant field density=ρ
Δρ = ρs-ρ
2bo
Pollutant first appears
at sea surface, x=0,
flux=qo
2b(x), width of field
Δy
A
Lateral spread
x
SECTION A-A
h(x), average depth of
pollution field
Uf
Pollution field, ρ
Uf
Surrounding
ocean, ρs
Idealized cross-section
Possible actual shape
of field cross-section
SECTION A-A

Figure 25. Interpretation of the Larsen and Sorensen Model for pure gravitational spread.

pollution field. By conservation of mechanical energy, the velocity of spread of the pollution front, U_f, is given by

$$U_f = \alpha_3 \sqrt{2g\bar{h}(x) \frac{\Delta\rho}{\rho}}, \tag{171}$$

where α_3 is a constant on the order of 1, and $\Delta\rho$ is the density disparity between layers. Because

$$U_f = U_0 \frac{db}{dx},$$

$$b(x) = \left[b_0^{3/2} + \frac{3\alpha_3}{2} \sqrt{\frac{g\Delta\rho q_0}{\rho U_\infty^3}}\ x \right]^{2/3} \tag{172}$$

where $2b_0$ is the initial width of the pollution field. As an approximation, for $b_0 << b(x)$,

$$b(x) \simeq \left[\sqrt{\frac{9g\Delta\rho q_0}{4\rho U_\infty^3}}\ x \right]^{2/3} \tag{173}$$

Thus, equation (173) may be used to estimate the pollution field width, 2b, at downstream points, x. However, this equation was developed from a very crude model and considers only gravitational effects. For these reasons, this model must be considered only a very rough approximation of the actual dispersion process.

Akira Wada and colleagues in a series of Japanese papers (55, 56, 29), have taken a more rigorous and realistic approach in analyzing the recirculation of condenser cooling water for nuclear power stations located on certain bays in Japan. The approach taken by Wada is to consider the fundamental equations of momentum and energy,

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + 2e_{ijk}\Omega_j U_k = -\frac{1}{\rho_o}\frac{\partial P}{\partial x_i} + \left(\frac{\rho_\infty - \rho}{\rho_o} - 1\right) g \ i3$$

$$+ \frac{\partial}{\partial x_j}\left[\varepsilon_{\underline{ij}} \frac{\partial U_i}{\partial x_j}\right], \qquad (8)$$

with molecular diffusivity, ν, ignored, and

$$\frac{\partial T}{\partial t} + U_j \frac{\partial T}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\varepsilon_{Hj} \frac{\partial T}{\partial x_j}\right) + \frac{\dot{q}}{\rho_o C_p}, \qquad (9)$$

subject to certain assumptions, and carry out the solutions numerically. These assumptions are:

- steady flow;
- inertial terms $\left(U_j \frac{\partial U_i}{\partial x_j}\right)$ are small compared to turbulent transport and pressure terms;
- Coriolis forces are neglected;
- the motion is a two-dimensional lateral flow restricted to the region above the thermocline.

Equations (8) and (9) are then written in x,y,z coordinates as:

x-direction momentum,

$$\frac{\partial P}{\partial x} = \frac{\partial}{\partial x}\left(\varepsilon_x \frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(\varepsilon_y \frac{\partial U}{\partial y}\right), \qquad (174)$$

y-direction momentum,

$$\frac{\partial P}{\partial y} \quad \frac{\partial}{\partial x}\left(\varepsilon_x \frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\varepsilon_y \frac{\partial V}{\partial y}\right), \qquad (175)$$

Energy,

$$U \frac{\partial T}{\partial x} + V \frac{\partial T}{\partial y} = \frac{\partial}{\partial x} (\varepsilon_{Hx} \frac{\partial T}{\partial x}) + \frac{\partial}{\partial y} (\varepsilon_{Hy} \frac{\partial T}{\partial y})$$

$$+ \frac{\partial}{\partial z} (\varepsilon_{Hz} \frac{\partial T}{\partial z}) + \frac{Q_o}{\rho C_p \bar{h}} , \tag{176}$$

where Q_o is net surface heat transfer,* C_p is specific heat, and $\bar{h}$ is the surface layer thickness. The quantity Q_o, the net heat exchange at the water surface, is composed of the net radiant exchange, evaporation, and convection. Note that in contrast to the single layer flow assumption for the equations of motion, heat transfer is treated in three dimensions.

By assuming the eddy diffusivities of momentum, ε_x and ε_y, to be constant, and by cross-differentiation, equations (174) and (175) become:

$$\frac{\partial^2 P}{\partial x \partial y} = \varepsilon_x \frac{\partial}{\partial y} \left(\frac{\partial^2 U}{\partial x^2}\right) + \varepsilon_y \frac{\partial}{\partial y} \left(\frac{\partial^2 U}{\partial y^2}\right) \tag{177}$$

$$\frac{\partial^2 P}{\partial x \partial y} = \varepsilon_x \frac{\partial}{\partial x} \left(\frac{\partial^2 V}{\partial x^2}\right) + \varepsilon_y \frac{\partial}{\partial x} \left(\frac{\partial^2 V}{\partial y^2}\right). \tag{178}$$

Now a stream function, ψ, may be defined such that

$$U = \frac{\partial \psi}{\partial y} \tag{179}$$

and

$$V = - \frac{\partial \psi}{\partial x} , \tag{180}$$

---

* Strictly speaking, the term $Q_o/\rho C_p \bar{h}$ should not be included in equation (176) since this term is a boundary condition and should be treated as such.

which automatically satisfy the continuity equation,

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0. \tag{181}$$

By subtracting (178) from (177), the terms involving pressure are eliminated. By replacing U and V with equations (179) and (180) in this result, an equation for horizontal motion in terms of the stream function may be written as

$$\frac{\partial^4 \psi}{\partial x^4} + (1+\delta)\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \delta\frac{\partial^4 \psi}{\partial y^4} = 0. \tag{182}$$

The quantity δ in equation (182) is the ratio $\varepsilon_y/\varepsilon_x$.

Wada solved equation (182) for the stream function, ψ, using finite difference techniques. Velocity components U and V were established using equations (179) and (180). Once the velocity components were determined, solution to equation (176) was found numerically, thus establishing the temperature distribution for the particular bay.

The approach taken seems to be a reasonable one. However, the hydraulics may be oversimplified by the single layer assumption. Also, it is not clear that all inertial terms in the equations of motion can be neglected - especially in the vicinity of the hot water source.

Hayashi and Shuto (20) also investigated the case of hot water spreading over a colder stagnant receiving water. Their approach was based on vertically integrated equations of motion and energy conservation, and similarity approximations concerning vertical velocity and temperature profiles. Except for the inclusion of vertical variations, these authors make the same flow field assumptions as did Wada et al. The governing equations become:

x-momentum,

$$\varepsilon_L\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\int_\zeta^d U dz - \frac{\partial}{\partial x}\int_\zeta^d P dz = 0; \qquad (183)$$

y-momentum,

$$\varepsilon_L\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\int_\zeta^d V dz - \frac{\partial}{\partial y}\int_\zeta^d P dz = 0; \qquad (184)$$

z-momentum,

$$P = \int_\zeta^z \rho g dz.$$

Here z is vertically downward, ζ is the ordinate at the water surface, d is the water depth, and ε_L is the lateral eddy viscosity which is assumed to be the same for both the x and y directions.

The vertically integrated continuity equation is

$$\frac{\partial}{\partial x}\int_\zeta^z U dz + \frac{\partial}{\partial y}\int_\zeta^z V dz = E'\sqrt{U^2+V^2}\ , \qquad (185)$$

where E' is the entrainment parameter (entrainment from lower depths, z direction).

The energy equation used by Hayashi and Shuto is written as

$$\frac{\partial}{\partial x}\int_\zeta^d (T-T_\infty)U dz + \frac{\partial}{\partial y}\int_\zeta^d (T-T_\infty)V dz = \frac{h_s}{\rho_o C_p}(T_s-T_\infty). \qquad (186)$$

In this equation T_∞ and T_s are the temperatures of the receiving water and hot water surface, respectively, and h_s is an overall heat transfer coefficient at the air-sea interface. Note that turbulent heat transport is not accounted for.

According to Hayashi and Shuto, at low values of Fr_J (the length term in the densimetric Froude number is based on the thickness of the thermal layer), the entrainment parameter is essentially zero, so that continuity is given by

$$\frac{\partial}{\partial x}\int_{\zeta}^{z} U dz + \frac{\partial}{\partial y}\int_{\zeta}^{z} V dz = 0. \tag{187}$$

Now a stream function may be defined such that

$$\alpha' hU = \frac{\partial\psi}{\partial y} \tag{188}$$

and $$\alpha' hV = -\frac{\partial\psi}{\partial x}, \tag{189}$$

where $\alpha'$is a parameter related to the vertical velocity profile and h is the thickness of the layer of motion.

It is now possible to arrive at equation (182) with $\delta=1(\varepsilon_x/\varepsilon_y=1)$, so that the stream function is given by

$$\nabla^4\psi = 0 \tag{190}$$

where ∇^4 is the biharmonic operator. For a simple geometry the above equation may be solved analytically. Hayashi and Shuto assumed the flow was issuing to a semi-infinite medium, as shown in Figure 26. For this geometry,

$$\psi = \frac{1}{\pi}\frac{Q}{2l}\left[(y+1)\tan^{-1}\left(\frac{y+1}{x}\right)-(y-1)\tan^{-1}\left(\frac{y-1}{x}\right)\right]. \tag{191}$$

From this equation, velocities U and V may be obtained from the

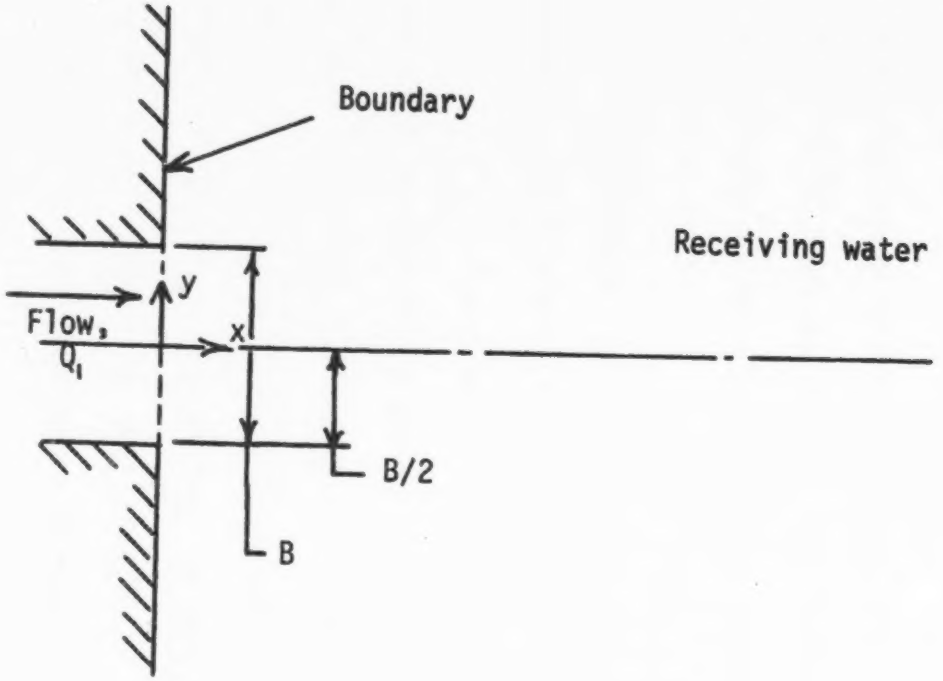
Boundary
Receiving water
y
Flow,
Q_1
x
B/2
B

Figure 26. Planar view of flow issuing to a semi-infinite medium.

definition of the stream function and applied in the energy equation to arrive at the temperature distribution,

$$T=T_{\infty}+(T_J-T_{\infty})\exp\left[-\frac{h_s}{\rho C_p}\frac{\alpha'}{\beta'}\frac{\pi}{4Q/B^2}\left\{1+\left(\frac{y}{x}\right)^2\right\}^2\left(\frac{x}{B'}\right)^2\right]. \qquad (192)$$

The quantity β' is a profile parameter associated with the vertical temperature profile. The authors assume that $\alpha'/\beta' = 1$, which is apparently to state that the temperature and velocity profiles are the same.

The preceding analysis is a very rough approximation of real conditions. As presented in the reference cited, there was poor agreement between theory and experiment.

The dynamic behavior of the drifting flow field is very complicated when gravitation forces are acting in conjunction with turbulent mixing and air-sea interactions, to disperse the pollutant. Reliable analytical or semi-empirical models are nonexistent for this case, and it appears that numerical simulation of the flow field may be the most reasonable approach to this problem.

NUMERICAL TECHNIQUES

Advances in computer technology during the past few years have opened a new frontier in solving partial differential equations such as those describing the motion and diffusion of pollutants in coastal waters. Before the age of high-speed computers, numerical techniques which had been set down in theory some years previous, were impractical to carry out because of the monumental amount of calculation involved. But today such techniques as finite differences, finite elements, and the Monte Carlo method are indeed practical and are being applied to a wide variety of practical engineering problems. In some cases these methods are so successful that they are used to save engineering time in spite of known analytic solutions.

The most attractive feature of numerical solutions is that many complexities associated with real physical phenomena do not have to be ignored as in the case of analytical solutions. Complex boundary geometry and boundary conditions can be accounted for and non-linear effects may be incorporated in the analysis. On the other hand, numerical solutions are not without drawbacks. Such problems as computation time and numerical stability in some cases negate the attractiveness of the numerical approach. Nevertheless, the numerical solution is a practical alternative to the oversimplified analytical solution.

The most complete treatment of numerical modeling would include the following considerations:

- time dependence,
- computation of velocity and diffusion in three-space,
- interaction between the flow field and pollutant field,
- variable eddy diffusivity coefficients, and
- arbitrary boundary conditions such as irregular solid boundaries, heat transfer and wind at the sea surface, and variable currents and water levels at the flow boundaries.

At present, there is no computer program available which includes all of the above complications. Typically, computations are carried out in two space dimensions, usually the horizontal plane. However, diffusion computations are in some cases extended to three dimensions. For instance, Wada (55) computed the velocity field for certain bays in Japan, using the two-dimensional hydrodynamics equations, but considered heat transfer in three dimensions.

If one is to consider interactions between the flow field and density distribution, which may be very important during initial dilution, three space dimensions must be considered unless the region of interest has symmetry to the extent that one space variable may be realistically eliminated. In coastal situations, symmetry is usually lacking because of irregular currents and boundaries. Wada (56) considered a horizontal shoreline discharge of hot water which included buoyancy interaction. However, the analysis was carried out in two dimensions, vertical and parallel to the discharge. Since no variation was considered parallel to the shoreline, the numerical model was not realistic in this respect. On the other hand, the analysis demonstrated significant differences in the flow and temperature fields between the cases of interaction and no interaction.

For the drift flow regime lateral density gradients caused by a contaminant are usually quite small. Thus, it is typical to neglect the interaction between buoyancy and flow and thus analyze the hydraulic problem separate from the pollutant diffusion. Velocity distributions calculated from hydraulic analysis or estimated from field measurements are then used as known input to compute convective terms in the transport equation.

Numerical Models

A considerable amount of work has appeared in the literature in recent years dealing with numerical methods for solving the equations of motion and diffusion ((5), (17), (49), (50),(53), and (57)). However, most of this work deals with specific cases and is not directly suited for coastal pollution problems. At this time there are only a few programs published and available for coastal hydraulic computation, and there is no program generally available as a complete package for predicting both coastal hydraulics and dispersion.

The remainder of this section will be confined to a brief discussion of published computer programs which under certain conditions are suitable for predicting coastal hydraulics and/or dispersion in the drift flow regime.

Coastal Hydraulics - Leendertse's Model

The numerical model developed by Leendertse (27) is based on a finite difference representation of the vertically integrated equations of motion and continuity. The model is specifically designed for tidal computations along a coast having arbitrary geometry and bottom topography. An operational computer program is listed in reference (27) along with input instructions. Dronkers (13) gives a review of the numerical model.

Numerical modeling is based on the following partial differential equations:

Motion.

$$\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} - f_c V + g \frac{\partial \zeta}{\partial x} + g \frac{U(U^2+V^2)^{1/2}}{C_h{}^2(H+\zeta)} = F^{(x)} , \tag{193}$$

$$\frac{\partial V}{\partial t} + U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} + f_c U + g \frac{\partial \zeta}{\partial y} + g \frac{V(U^2+V^2)^{1/2}}{C_h{}^2(H+\zeta)} = F^{(y)} \tag{194}$$

where U and V are the vertically averaged velocity components u and v defined by

$$U = \frac{1}{H+\zeta} \int_{-h}^{\zeta} u dz$$

$$V = \frac{1}{H+\zeta} \int_{-h}^{\zeta} v dz.$$

$F^{(x)}$ and $F^{(y)}$ are forcing functions from wind stress and barometric pressure fluctuations, C_h is the Chezy coefficient, and ζ is the fluctuating water height measured from a reference. This reference plane is a distance H from the ocean bottom.

Continuity. The vertically integrated continuity equation leads to the following expression for the fluctuating water level ζ:

$$\frac{\partial\zeta}{\partial t} + \frac{\partial\{(H+\zeta)U\}}{\partial x} + \frac{\partial\{(H+\zeta)V\}}{\partial y} = 0. \qquad (195)$$

The method of solution is what Leendertse terms a multi-operational technique. This technique is very closely associated with the alternating direction method for two-dimensional partial differential equations first discussed by Peaceman and Rachford (40a) concerning the heat equation. Details of the solution technique will not be discussed here; however, the interested reader may consult Leendertse(27) or Dronkers (13).

Leendertse has carried out detailed studies of the method stability and has ascertained that the model is unconditionally numerically stable. However, solution accuracy deteriorates as the time step becomes large.

Input to the computation program includes system geometry-flow boundary water levels as a function of time, and Chezy coefficients. Output includes water level and vertically integrated velocity at each grid point in the system as a function of time.

Although the vertically integrated model may not yield the velocity field detail that is required for studying localized dispersion, it does provide flow information such that gross transport and dispersion may be estimated.

Dispersion

Three available computer codes are briefly discussed here:

1) NRDL codes (23) based on the mathematical solution of the Carter-Okubo (10a) model;

2) Tetra-Tech code (25) used for studying long-term transport;

3) New York University, (30), based on the Monte Carlo method.

In each of the above codes the velocity field must be established independently, and each has limited application.

Naval Radiological Defense Laboratory Codes. The Naval Radiological Defense Laboratory (NRDL) has developed computer codes for estimating radioactivity levels which result from instantaneous and continuous point releases in the ocean. These models are based on the analytical solution of the Carter-Okubo shear flow model (10a):

$$\frac{\partial C}{\partial t} + (U_o - \Omega'_y y - \Omega'_z z) \frac{\partial C}{\partial x}$$

$$= \varepsilon_{Hx} \frac{\partial^2 C}{\partial x^2} + \varepsilon_{Hy} \frac{\partial^2 C}{\partial y^2} + \varepsilon_{Hz} \frac{\partial^2 C}{\partial z^2} . \qquad (196)$$

The terms Ω'_y and Ω'_z are velocity gradients in the y and z direction and U_o is the mean current. These terms are assumed constant for a given solution to the above transport equation.

The computer programs calculate the dimensions and volume, and amount of contaminant, in a dispersing patch caused by a point or continuous release. Program input includes the empirical constants U_o, Ω'_y, Ω'_z, ε_x, ε_y and ε_z. This program is not applicable where finite boundaries, such as the water surface and coastline, affect dispersion.

Tetra-Tech Code. Koh and Fan (25) have developed a computer program for estimating long-term transport for a pool of radioactive debris submerged in the ocean, a problem similar to the one investigated by NRDL, described above. However, Koh and Fan, instead of using the analytical solution of the Carter-Okubo transport equation as was done in the NRDL programs, solve the basic diffusion equation,

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} + V\frac{\partial C}{\partial y} = \frac{\partial}{\partial x}\left(\varepsilon_{Mx}\frac{\partial C}{\partial x}\right)$$

$$+ \frac{\partial}{\partial y}\left(\varepsilon_{My}\frac{\partial C}{\partial y}\right) + \frac{\partial}{\partial z}\left(\varepsilon_{Mz}\frac{\partial C}{\partial z}\right), \qquad (197)$$

by the "method of moments." The above equation does not include vertical convective transport of the pollutant.

Details of the method of moments are not discussed here (e.g., see reference 48), but it is a technique of lumping quantities in a horizontal plane so that the concentration moments are a function of z and t alone. This method then greatly reduces the amount of time necessary for numerical solutions to the transport equation.

The computer program application is limited to a submerged pool with no coastline effects. Horizontal velocities are assumed constant in a lateral plane, but may vary with depth. Velocities must be supplied to the program as input. Also, no transport across the water surface is allowed.

New York University Code. Mehr (30) has developed a dispersion code based on the Monte Carlo method using random numbers. At this time full detail concerning the program and solution technique has not been published, but may be obtained from Mehr.

According to Mehr, the program will predict contaminant distributions for a two-dimensional horizontal flow system having laterally irregular solid boundaries. As in the case of previously discussed dispersion codes, the velocity field must be supplied by the program user. The general idea of the method is that a large number of contaminant particles are tagged. Beginning with an initial distribution of these tagged particles, they are convected through the flow field according to the velocity distribution, and diffused according to random number components. Solid boundaries reflect these tagged particles and crossing over flow boundaries causes the particles to be lost from the system. The program will accommodate time dependent particle sources; hence, an outfall may be simulated.

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NOMENCLATURE

The following notations are used unless otherwise specified in the text.

a	Molecular thermal diffusivity
b	Characteristic plume radius
$b(x)$	Half-width of pollution field
C	Concentration of pollutant
C_d	Coefficient of drag
C_h	Chezy coefficient
C_m	Concentration of pollutant at centerline
C_p	Specific heat
D	Diameter of discharge port
$\mathcal{D}$	Molecular mass diffusivity
E	Cube of centerline velocity (U_m^3)
e_{ijk}	Permutation tensor (third order)
$\vec{e}_r$	Unit vector, radial
$\vec{e}_x$	Unit vector, east
$\vec{e}_y$	Unit vector, north
$\vec{e}_z$	Unit vector, vertical
$\vec{e}_\theta$	Unit vector, angular
erf	Error function

f	Function
f_c	Coriolis parameter
$F^{(\ )}$	Forcing function
F_d	Drag force
Fr	Densimetric Froude number
Fr_J	Densimetric Froude number at discharge port
g	Gravitational constant
G	Linear stratification parameter
$\bar{h}$	Average thickness of pollution field
h_s	Heat transfer coefficient
H	Water depth
k	Proportionality constant
K	Entrainment parameter for momentum
L	Lateral scale for drifting plume
l	Width of pollution field
$\dot{m}$	Mass species loss or production per unit volume
M	Instantaneous source strength
M_e	Momentum of plume at beginning of established flow
M_j	Momentum at orifice
P	Pressure
$\dot{q}$	Volumetric heat generation
q_o	Volumetric flow rate of pollutant
Q	Volume flow rate
Q_o	Surface heat transfer
Q'	Volumetric entrainment

r	Radial coordinate
R	Product $(U_m \Delta_m)$
R_b	Radius of plume boundary
R_c	Characteristic radial scale for plume
R_{ij}	Reynolds stress tensor
s	Axis for curved plume
S	Dilution (1/C)
S'	Source strength
S_e	Length to established flow, curved plume
S_m	Centerline dilution
t	Time
t'	Length of time for averaging turbulent quantities
T	Temperature
T_o	Reference temperature $(T_\infty(o))$
T_m	Temperature at plume centerline
$T_\infty(z)$	Temperature of environment
u_i	Fluctuating velocity along $i\underline{th}$ coordinate (description of turbulent flow)
U	Velocity (x-direction)
U_i	Velocity along $i\underline{th}$ coordinate
U_m	Centerline velocity (horizontal and inclined plumes)
U_r	Velocity (radial direction)
U_∞	Environmental velocity, x-component
U_θ	Velocity (angular)

U_o, U_J	Orifice velocity
v	Volume
V	Velocity (y-direction)
V_∞	Environmental velocity, y-component
W	Velocity (z-direction)
W_J	Orifice velocity
W_m	Vertical velocity at centerline
W_∞	Environmental velocity, z-component
x	Space coordinate
x_i	$i^{\underline{th}}$ space coordinate
y	Space coordinate
z	Space coordinate, vertical
z_e, x_e	Length for flow establishment
z_{max}	Maximum height of rise at plume centerline

Greek

α	Entrainment coefficient
β	Temperature coefficient
Δ	Density disparity parameter
$\Delta\gamma$	Buoyant force
δ	Ratio, ϵ_x/ϵ_y
δ_{ij}	Kronecker delta function (second order tensor)
ζ	Water level
ϵ_{ij}	Eddy momentum diffusivity tensor (second order)
ϵ_{Hj}	Eddy thermal diffusivity tensor (first order)

ε_j	Eddy momentum diffusivity
Θ	Temperature disparity
θ	Angular coordinate
κ	Thermal conductivity
λ	Entrainment coefficient for heat or matter (Morton, Fan)
μ	Entrainment parameter for heat or matter
ν	Molecular kinematic viscosity
ρ	Density
ρ_J	Density of pollutant at orifice
ρ_m	Plume centerline density
ρ_o	Density of environment at orifice
ρ_s	Density of environment at surface
$\rho_\infty(z)$	Density of environment
π	Pi, 3.141
τ	Fluid stress tensor (second order)
Φ	Viscous dissipation term in energy equation
ϕ	Latitude
$\phi(x_i)$	Function
$\Psi(x_i)$	Function
ψ	Stream function
Ω'	Velocity gradient

DEPARTMENT OF THE INTERIOR

In its assigned function as the Nation's principal natural resource agency, the Department of the Interior bears a special obligation to assure that our expendable resources are conserved, that renewable resources are managed to produce optimum yields, and that all resources contribute their full measure to the progress, prosperity, and security of America, now and in the future.